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Compact Quantum Groups and Their Representation Categories

Sergey NESHVEYEV & Lars TUSET

20

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Sergey Neshveyev

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PREFACE

The term 'quantum group' was popularized in the 1980s and, in fact, does not have a precise meaning. The closely related, and rigorously defined, notion of a Hopf algebra appeared much earlier, in the 1950s. It has its origin in a work by Hopf [**41**] from 1941 on algebraic topology, who observed that the cohomology ring of a compact group G has a homomorphism $H^*(G) \to H^*(G) \otimes H^*(G)$. A related, and even more elementary, example of such structure is the following: for a finite group G and the algebra $C(G)$ of functions on G with pointwise multiplication we can define a homomorphism, called comultiplication,

$$\Delta\colon C(G) \to C(G) \otimes C(G) = C(G \times G) \ \text{ by } \ \Delta(f)(g, h) = f(gh).$$

What is important, is that the pair $(C(G), \Delta)$ contains complete information about the group G: the spectrum of the algebra $C(G)$ is G, and the comultiplication Δ allows us to recover the group law. We refer the reader to [**1**] for a thorough discussion of the origins of the theory of Hopf algebras. The part of the story that is particularly relevant for us starts in the early 1960s with a work by Kac [**47**]. His idea was to develop a duality theory that generalizes Pontryagin duality for abelian locally compact groups. Such a generalization for compact groups had already been obtained by Tannaka [**77**] and Krein [**54**], but even in that case it was not entirely satisfactory in the sense that the dual of a compact group G was an object of a quite different nature, the category of finite dimensional representations of G concretely realized as a category of vector spaces. Kac's idea was to describe both a locally compact group and its dual using von Neumann algebras with comultiplication satisfying certain properties, and this way obtain a self-dual category. Such a theory, nowadays called the theory of Kac algebras, was finally developed in the 1970s by Kac-Vainerman and Enock-Schwartz, see [**29**].

Being a significant technical achievement, the theory of Kac algebras nevertheless suffered from the lack of interesting examples that were not of group origin, that is, were neither algebras of functions nor their duals, group algebras. For similar reasons the general theory of Hopf algebras remained at that time a small branch of algebra. The situation changed drastically in the middle 1980s, when Jimbo [**44**] and Drinfeld [**26**] introduced new Hopf algebras by deforming universal enveloping algebras of semisimple Lie groups. Working in the formal deformation setting Drinfeld also introduced their dual objects, deformations of the Hopf algebras of regular functions on semisimple Lie groups. He suggested the term 'quantum groups' for Hopf algebras related to these constructions. In the analytic, non-formal, setting the quantized algebra of functions on $SU(2)$ was then studied in detail by Vaksman and Soibelman [**82**]. Simultaneously, and independently of Drinfeld-Vaksman-Soibelman, a deformation of the algebra of continuous functions on $SU(2)$ was defined by Woronowicz [**95**]. Remarkably, he arrived at exactly the same definition.

Following the foundational works by Drinfeld, Jimbo, Soibelman, Vaksman and Woronowicz, the theory of quantum groups saw several years of explosive growth, apparently unprecedented in the history of mathematics, with ground-breaking applications to knot theory, topology of 3-manifolds and conformal field theory [**80**]. This was entwined with the development of noncommutative geometry by Connes, the free probability theory by Voiculescu and the Jones theory of subfactors. Since then quantum group theory has developed in several directions and by now there is probably no single expert who has a firm grasp of all of its aspects.

In this book we take the analytic point of view, meaning that we work with algebras of, preferably bounded, operators on Hilbert spaces. For an introduction to quantum groups from the purely algebraic side see e.g., [**18**]. According to the standard mantra of noncommutative geometry, an algebra of operators should be thought of as an algebra of functions on a noncommutative locally compact space, with C^*-algebras playing the role of continuous functions and von Neumann algebras playing the role of measurable functions. From this perspective a quantum group is a C^*-/von Neumann algebra with some additional structure making the noncommutative space a group-like object. Kac algebras give an example of such structure, but as it turned out their class is too narrow to accommodate the objects arising from Drinfeld-Jimbo deformations. A sufficiently broad theory was developed first in the compact case by Woronowicz [**97**], and then in the general, significantly more complicated, locally compact case by Kustermans-Vaes [**55**] and Masuda-Nakagami-Woronowicz [**63**]. No theory is complete without interesting examples, and here there are plenty of them. In addition to examples arising from Drinfeld-Jimbo deformations, there is a large class of quantum groups defined as symmetries of noncommutative spaces. This line of research was initiated by Wang [**89, 90**] and has been extensively pursued by Banica and his collaborators, see e.g., [**8, 9**]. A related idea is to define quantum isometries of noncommutative Riemannian manifolds, recently suggested by Goswami and Bhowmick [**37, 12**].

The goal of this short book is to introduce the reader to this beautiful area of mathematics, concentrating on the technically easier compact case and emphasizing the role of the categorical point of view in constructing and analyzing concrete examples. Specifically, the first two chapters, occupying approximately 2/3 of the book, contain a general theory of compact quantum groups together with some of the most famous examples. Having mastered the material in these chapters, the reader will hopefully be well prepared for a more thorough study of any of the topics we mentioned above. The next two chapters are motivated by our own interests in noncommutative geometry of quantum groups and concentrate on certain aspects of the structure of Drinfeld-Jimbo deformations. The general theme of these chapters is the Drinfeld-Kohno theorem, which is one of the most famous results in the whole theory of quantum groups, presented from the analytic point of view together with its operator algebraic ramifications. Each section is supplied with a list of references. We try to give references to original papers, where the results of a particular section and/or some related results have appeared. The literature on quantum groups is vast, so some omissions are unavoidable, and the references are meant to be pointers to the literature rather than exhaustive bibliographies on a particular subject.

We tried to make the exposition reasonably self-contained, but certain prerequisites are of course assumed. The book is first of all intended for students specializing in operator algebras, so we assume that the reader has at least taken a basic course in C*-algebras as e.g., covered in Murphy [**65**]. The reader should also have a minimal knowledge of semisimple Lie groups, see e.g., Part II in Bump [**17**], without which it is difficult to fully understand Drinfeld-Jimbo deformations. Finally, it is beneficial to have some acquaintance with category theory. Although we give all the necessary definitions in the text (apart from the most basic ones, for those see e.g., the first chapter in Mac Lane [**61**]), the reader who sees them for the first time will have to work harder to follow the arguments.

Let us say a few words about notation.

We denote by the same symbol $\otimes$ all kinds of tensor products, the exact meaning should be clear from the context: for spaces with no topology this denotes the usual tensor product over $\mathbb{C}$, for Hilbert spaces - the tensor product of Hilbert spaces (that is, the completion of the algebraic tensor product with respect to the obvious scalar product), for C*-algebras - the *minimal* tensor product.

For vector spaces H_1 and H_2 with no topology we denote by $B(H_1, H_2)$ the space of linear operators $H_1 \to H_2$. If H_1 and H_2 are Hilbert spaces, then the same symbol $B(H_1, H_2)$ denotes the space of bounded linear operators. We write $B(H)$ instead of $B(H, H)$.

If $\mathscr{A}$ is a vector space with no topology, then $\mathscr{A}^*$ denotes the space of all linear functionals on $\mathscr{A}$. For topological vector spaces the same symbol denotes the space of all continuous linear functionals.

The symbol ι denotes the identity map.

In order to simplify long complicated expressions, we omit the symbol $\circ$ for the composition of maps, as well as use brackets only for arguments, but not for maps. Thus we write $ST(x)$ instead of $(S \circ T)(x)$.

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Sergey Neshveyev & Lars Tuset
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CHAPTER 1

COMPACT QUANTUM GROUPS

This chapter contains fundamentals of the theory of compact quantum groups. After some basic definitions we present several nontrivial examples of such objects. The core of the chapter is the representation theory. We first develop it in finite dimensions, and then move to infinite dimensional representations, which unavoidably requires a bit better acquaintance with operator algebras. Overall the theory is very similar to the representation theory of compact groups. The additional technical difficulties arise from modular properties of the quantum analogue of Haar measure. Among other topics we discuss the dual picture of discrete quantum groups, which plays an important role in the subsequent chapters.

1.1. DEFINITION AND FIRST EXAMPLES

According to the philosophy of noncommutative geometry, unital C*-algebras should be thought of as algebras of continuous functions on noncommutative, or quantum, compact spaces. We want to define a 'group structure' on such spaces. As we will gradually convince ourselves, the following definition makes a lot of sense.

Definition 1.1.1. — (Woronowicz) A compact quantum group is a pair (A,Δ), where A is a unital C*-algebra and $\Delta\colon A\to A\otimes A$ is a unital $*$-homomorphism, called comultiplication, such that

(i) $(\Delta\otimes\iota)\Delta=(\iota\otimes\Delta)\Delta$ as homomorphisms $A\to A\otimes A\otimes A$ (coassociativity);

(ii) the spaces $(A\otimes 1)\Delta(A)=\operatorname{span}\{(a\otimes 1)\Delta(b)\mid a,b\in A\}$ and $(1\otimes A)\Delta(A)$ are dense in $A\otimes A$ (cancellation property).

We remind that by the tensor product of C*-algebras we always mean the minimal tensor product.

Example 1.1.2. — Let G be a compact group. Take A to be the C*-algebra $C(G)$ of continuous functions on G. Then $A\otimes A = C(G\times G)$, so we can define Δ by

$$\Delta(f)(g,h) = f(gh) \ \text{ for all } \ g,h\in G.$$

Coassociativity of Δ follows from associativity of the product in G. To see that the cancellation property holds, note that $(A\otimes 1)\Delta(A)$ is the unital $*$-subalgebra of $C(G\times G)$ spanned by all functions of the form $(g,h)\mapsto f_1(g)f_2(gh)$. Since such functions separate points of G, the $*$-algebra $(A\otimes 1)\Delta(A)$ is dense in $C(G\times G)$ by the Stone-Weierstrass theorem.

Any compact quantum group (A,Δ) with abelian A is of this form. Indeed, by the Gelfand theorem, $A = C(G)$ for a compact space G. Then, since $A\otimes A = C(G\times G)$, the unital $*$-homomorphism Δ is defined by a continuous map $G\times G\to G$. Coassociativity means that

$$f((gh)k) = f(g(hk)) \ \text{ for all } \ f\in C(G),$$

whence $(gh)k = g(hk)$, so G is a compact semigroup. If $gh = gk$, then $f_1(g)f_2(gh) = f_1(g)f_2(gk)$ for all $f_1,f_2 \in C(G)$. By the cancellation property the functions of the form $(g',h') \mapsto f_1(g')f_2(g'h')$ span a dense subspace of $C(G\times G)$. It follows that $f(g,h) = f(g,k)$ for all $f\in C(G\times G)$, whence $h = k$. Similarly, if $hg = kg$, then $h = k$. Thus G is a semigroup with cancellation.

It remains to show that any compact semigroup with cancellation is a group. Let us first show that G has a (necessarily unique) unit element. For this take any element $h\in G$ and let H be the closed subsemigroup of G generated by h. It is clearly abelian. If $I_1, I_2\subset H$ are ideals, then $I_1\cap I_2\supset I_1I_2 = I_2I_1$, so by compactness the intersection I_h of all closed ideals in H is a nonempty closed ideal. Then $kI_h = I_h$ for any $k\in H$, because $kI_h\subset I_h$ and kI_h is a closed ideal in H. In particular, taking $k\in I_h$ we can find $e\in I_h$ such that $ke = k$. Then $keg = kg$ for any $g\in G$, and by cancellation we get $eg = g$. Similarly $ge = g$. Thus e is a unit in G.

In order to see that every element h in G is invertible, note that by the above argument we have $e\in I_h = hI_h$. Hence there exists $k\in I_h$ such that $kh = hk = e$.

It remains to show that the map $g\mapsto g^{-1}$ is continuous. For this, consider the map $G\times G\to G\times G$, $(g,h)\mapsto (g,gh)$. It is continuous and bijective. Since the space $G\times G$ is compact, it follows that the inverse map, $(g,h)\mapsto (g,g^{-1}h)$, is also continuous. Hence the map $g\mapsto g^{-1}$ is continuous. $\square$

In view of this example, for any compact quantum group $G = (A,\Delta)$ we write $C(G)$ for A.

As we saw in the above example, when G is a genuine compact group, the cancellation property is used to prove the existence of inverse. One may wonder why one does not try to encode in the definition of a compact quantum group the properties of the map $f \mapsto \check{f}$, where $\check{f}(g) = f(g^{-1})$, instead of introducing the cancellation property. The reason is that, as we will see later, the analogue of this map in the quantum setting is unbounded in most interesting cases.

Example 1.1.3. — Let Γ be a discrete group. Consider the left regular representation Γ on $\ell^2(\Gamma)$, which maps $\gamma \in \Gamma$ into the operator λ_γ defined by $\lambda_\gamma \delta_{\gamma'} = \delta_{\gamma\gamma'}$. The reduced C*-algebra $C^*_r(\Gamma)$ of Γ is by definition the closed linear span of the operators λ_γ, $\gamma \in \Gamma$. Define a compact quantum group $G = \hat{\Gamma}$ as follows:

$$C(G) = C^*_r(\Gamma), \quad \Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma.$$

The comultiplication Δ is cocommutative, in the sense that $\Delta = \Delta^{\mathrm{op}}$, where $\Delta^{\mathrm{op}} = \Sigma\Delta$ and $\Sigma\colon C^*_r(\Gamma) \otimes C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ is the flip.

Instead of $C^*_r(\Gamma)$ we could also consider the full group C*-algebra $C^*(\Gamma)$. Any compact quantum group with cocommutative comultiplication sits between $C^*_r(\Gamma)$ and $C^*(\Gamma)$ for a uniquely defined discrete group Γ. Rather than proving this, we will confine ourselves to showing how Γ can be recovered from $(C^*_r(\Gamma), \Delta)$:

$$\{\lambda_\gamma \mid \gamma \in \Gamma\} = \{a \in C^*_r(\Gamma) \mid \Delta(a) = a \otimes a,\ a \neq 0\}.$$

To see this, assume $a \in C^*_r(\Gamma)$, $a \neq 0$, is such that $\Delta(a) = a \otimes a$. Let τ be the canonical trace on $C^*_r(\Gamma)$, so $\tau(\lambda_\gamma) = 0$ for $\gamma \neq e$. Since τ is faithful, replacing a by $\lambda_\gamma a$ for some γ we may assume that $\tau(a) \neq 0$. Consider the completely positive map $E\colon C^*_r(\Gamma) \to C^*_r(\Gamma)$ defined by $E = (\iota \otimes \tau)\Delta$. We have $E(a) = \tau(a)a$ and $E(\lambda_\gamma) = 0$ for $\gamma \neq e$. Since a can be approximated in norm by linear combinations of elements λ_γ, we see that $\tau(a)a$ can be approximated arbitrary well by a multiple of λ_e, so a is a scalar. As $\Delta(a) = a \otimes a$, we thus get $a = 1$. $\square$

To give examples of compact quantum groups that are neither commutative nor cocommutative, we will need a sufficient condition for the cancellation property.

Proposition 1.1.4. — *Assume A is a unital C^*-algebra generated by elements u_{ij}, $1 \leq i, j \leq n$, such that the matrices $(u_{ij})_{i,j}$ and $(u^*_{ij})_{i,j}$ are invertible in $\mathrm{Mat}_n(A)$, and $\Delta\colon A \to A \otimes A$ is a unital $*$-homomorphism such that*

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Then (A, Δ) is a compact quantum group.

Proof. — The coassociativity of Δ is immediate. To prove that $\Delta(A)(1 \otimes A)$ is dense in $A \otimes A$, it suffices to show that the space

$$B = \left\{ a \in A : a \otimes 1 = \sum_i \Delta(x_i)(1 \otimes y_i) \text{ for some } x_i, y_i \right\}$$

is dense in A. Note that B is an algebra, as if $a \otimes 1 = \sum_i \Delta(x_i)(1 \otimes y_i)$ and $b \otimes 1 = \sum_j \Delta(x'_j)(1 \otimes y'_j)$, then

$$ab \otimes 1 = \sum_i \Delta(x_i)(b \otimes 1)(1 \otimes y_i) = \sum_{i,j} \Delta(x_i x'_j)(1 \otimes y'_j y_i).$$

Let $V = (v_{ij})_{i,j}$ be the inverse of $(u_{ij})_{i,j}$. Then

$$\sum_{j=1}^{n} \Delta(u_{ij})(1 \otimes v_{jk}) = \sum_{j,l=1}^{n} u_{il} \otimes u_{lj} v_{jk} = u_{ik} \otimes 1,$$

so $u_{ik} \in B$. Similarly $u_{ik}^* \in B$. Hence B is dense in A, and therefore $(1 \otimes A)\Delta(A)$ is dense in $A \otimes A$. Similarly, $(A \otimes 1)\Delta(A)$ is dense in $A \otimes A$. □

Compact quantum groups satisfying the assumptions of the above proposition are called compact matrix pseudogroups.

We can now give several examples of compact quantum groups. In all these examples the C*-algebra $C(G)$ will be defined as a universal C*-algebra generated by elements satisfying certain relations. In general one should be careful with such constructions, since not every $*$-algebra admits a C*-enveloping algebra. But in all our examples the matrix formed by the generators will be assumed to be unitary. In this case the norm of every generator is not larger than 1 in every $*$-representation by bounded operators on a Hilbert space, so the C*-enveloping algebra is well-defined. What is, however, more difficult to see in many cases is that the C*-algebras we will define are sufficiently large. The right tools for showing this will be developed in the next chapter.

Example 1.1.5. — (Quantum $SU(2)$ group)

Assume $q \in [-1, 1]$, $q \neq 0$. The quantum group $SU_q(2)$ is defined as follows. The algebra $C(SU_q(2))$ is the universal unital C*-algebra generated by elements α and γ such that $(u_{ij})_{i,j} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ is unitary. As one can easily check, this gives the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \ \ \alpha\alpha^* + q^2\gamma^*\gamma = 1, \ \ \gamma^*\gamma = \gamma\gamma^*, \ \ \alpha\gamma = q\gamma\alpha, \ \ \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication is defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj},$$

so

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Let $\lambda \in \mathbb{C}$ be such that $\lambda^2 = q$. Then

$$\begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha^* & -q\gamma \\ \gamma^* & \alpha \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Hence $(u_{ij}^*)_{i,j}$ is invertible. Therefore $SU_q(2)$ is a compact quantum group.

When $q=1$ we get the usual compact group $SU(2)$. Indeed, consider the functions $\tilde{u}_{ij}$ on $SU(2)$ defined by $\tilde{u}_{ij}(V)=V_{ij}$ for $V\in SU(2)$. Then by universality we have a unital $*$-homomorphism $\pi\colon C(SU_1(2))\to C(SU(2))$ such that $\pi(u_{ij})=\tilde{u}_{ij}$. This homomorphism is surjective by the Stone-Weierstrass theorem. Since $C(SU_1(2))$ is abelian, in order to see that π is injective it suffices to show that any character of $C(SU_1(2))$ factors through $C(SU(2))$. Assume χ is such a character. Then $\chi(U)\in SU(2)$. But this means that χ is the composition of the character $f\mapsto f(\chi(U))$ on $C(SU(2))$ with π. Thus π is an isomorphism. Finally, π respects comultiplication: the identity

$$\Delta(\tilde{u}_{ij})=\sum_k \tilde{u}_{ik}\otimes\tilde{u}_{kj}$$

is simply an equivalent way of writing matrix multiplication for elements of $SU(2)$.

The quantum groups $SU_q(2)$ for $q\neq 1$ can be thought of as deformations of $SU(2)$. We will make this statement a bit more precise later. □

Example 1.1.6. — (Free unitary quantum groups)

Let $F\in \mathrm{GL}_n(\mathbb{C})$, $n\geq 2$, be such that $\mathrm{Tr}(F^*F)=\mathrm{Tr}((F^*F)^{-1})$. Denote by $A_u(F)$ the universal C*-algebra generated by elements u_{ij}, $1\leq i,j\leq n$, such that

$$U=(u_{ij})_{i,j}\ \text{ and }\ FU^cF^{-1}\ \text{ are unitary,}$$

where $U^c=(u_{ij}^*)_{i,j}$. The comultiplication is defined by

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}.$$

We will use the same notation $A_u(F)$ for this compact quantum group and for the C*-algebra of continuous functions on it.

For $F=1$ the algebra $A_u(F)$ is denoted by $A_u(n)$. It is a 'liberation' of $C(U(n))$ in the sense that $C(U(n))$ satisfies the same relations as $A_u(n)$ plus commutativity. □

Example 1.1.7. — (Free orthogonal quantum groups)

Let $F\in \mathrm{GL}_n(\mathbb{C})$, $n\geq 2$, be such that $F\bar{F}=\pm 1$, where $\bar{F}$ is the matrix obtained from F by taking the complex conjugate of every entry. Define $A_o(F)$ as the universal C*-algebra generated by u_{ij}, $1\leq i,j\leq u$, such that

$$U=(u_{ij})_{i,j}\ \text{ is unitary and }\ U=FU^cF^{-1}.$$

The comultiplication is defined by

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}.$$

Again, we will use the same notation $A_o(F)$ for this compact quantum group and for the C*-algebra of continuous functions on it.

Note that we automatically have $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$. Therefore $A_o(F)$ is a quotient of $A_u(F)$.

Observe also that $SU_q(2)$ is an example of a free orthogonal group, with $F = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$, where λ is a square root of q.

Similarly to the previous example, for $F = 1$ the algebra $A_o(F)$ is denoted by $A_o(n)$. It is a liberation of $C(O(n;\mathbb{R}))$, meaning that $C(O(n;\mathbb{R}))$ satisfies the same relations as $A_o(n)$ plus commutativity. □

Example 1.1.8. — (Quantum permutation groups)

For $n \in \mathbb{N}$, denote by $A_s(n)$ the universal C*-algebra generated by elements u_{ij}, $1 \le i, j \le n$, such that

$$U = (u_{ij})_{i,j} \ \text{ is a 'magic unitary'},$$

meaning that U is unitary, all its entries u_{ij} are projections, and the sum of the entries in every row and column of U is equal to one. As before, the comultiplication is defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

We have a unital $*$-homomorphism $C(A_s(n)) \to C(S_n)$ respecting comultiplication, where S_n is the symmetric group, mapping u_{ij} into the characteristic function of the set $\{\sigma \in S_n \mid \sigma(i) = j\}$. As in the previous two examples, it is not difficult to check that $A_s(n)$ is a liberation of $C(S_n)$.

It can be shown that for $n = 1, 2, 3$ we have $A_s(n) = C(S_n)$, but for $n \ge 4$ the algebra $A_s(n)$ is noncommutative and infinite dimensional. □

References. — [**4**], [**8**], [**9**], [**29**], [**55**], [**63**], [**82**], [**87**], [**89**], [**90**], [**94**], [**95**], [**97**].

1.2. HAAR STATE

Let G be a compact quantum group. For bounded linear functionals ω_1 and ω_2 on $C(G)$ define their convolution by

$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2)\Delta.$$

When G is a genuine compact group, this gives the usual definition of convolution of measures on G. The Haar measure ν on G is characterized by the property that $\mu * \nu = \nu * \mu = \mu(G)\nu$ for any complex measure μ on G. The following theorem therefore extends the existence and uniqueness of a normalized Haar measure to the quantum setting.

Theorem 1.2.1. — *For any compact quantum group G, there exists a unique state h on $C(G)$ such that*

$$\omega * h = h * \omega = \omega(1)h \text{ for all } \omega \in C(G)^*.$$

Proof. — The uniqueness is clear. We will prove the existence in several steps.

Step 1. Let ω be a state on $C(G)$. Then there exists a state h on $C(G)$ such that $\omega*h = h*\omega = h$.

Indeed, take any weak* limit point of the states

$$\frac{1}{n}\sum_{k=1}^{n} \omega^{*k}.$$

Step 2. If $0 \le \nu \le \omega$ and $\omega * h = h * \omega = \omega(1)h$, then $\nu * h = h * \nu = \nu(1)h$.

We may assume that $\omega(1) = 1$. Fix $a \in C(G)$ and put $b = (\iota \otimes h)\Delta(a)$. Then

$$\begin{aligned}&(h\otimes\omega)((\Delta(b) - b\otimes 1)^*(\Delta(b) - b\otimes 1))\\ &\quad = (h*\omega)(b^*b) - (h\otimes\omega)(\Delta(b)^*(b\otimes 1)) - (h\otimes\omega)((b^*\otimes 1)\Delta(b)) + h(b^*b) = 0,\end{aligned}$$

since by coassociativity of Δ we have

$$(\iota\otimes\omega)\Delta(b) = (\iota\otimes\omega * h)\Delta(a) = (\iota\otimes h)\Delta(a) = b.$$

It follows that

$$(h\otimes\nu)((\Delta(b) - b\otimes 1)^*(\Delta(b) - b\otimes 1)) = 0.$$

By the Cauchy-Schwarz inequality we then get

$$(h\otimes\nu)((c\otimes 1)(\Delta(b) - b\otimes 1)) = 0 \text{ for all } c \in C(G).$$

Using this we compute:

$$\begin{aligned}(h\otimes\nu * h)((c\otimes 1)\Delta(a)) &= (h\otimes\nu)((c\otimes 1)\Delta(b))\\ &= (h\otimes\nu)(cb\otimes 1) = h(cb)\nu(1)\\ &= \nu(1)(h\otimes h)((c\otimes 1)\Delta(a)).\end{aligned}$$

Since $(C(G)\otimes 1)\Delta(C(G))$ is dense in $C(G)\otimes C(G)$, this implies $\nu * h = \nu(1)h$. Similarly, $h*\nu = \nu(1)h$.

End of proof. For a finite set $F = \{\omega_1, \dots, \omega_n\}$ of states on $C(G)$, take

$$\omega_F = \frac{1}{n}(\omega_1 + \cdots + \omega_n).$$

By Steps 1 and 2 we can find a state h_F such that $\omega_F * h_F = h_F * \omega_F = h_F$, hence $\omega_i * h_F = h_F * \omega_i = h_F$ for all i. Taking a weak* limit point of the states h_F as F increases, we get the required state h. □

The state h given by the above theorem is called the Haar state. An equivalent form of the defining property of h is

$$(\iota\otimes h)\Delta(a) = (h\otimes\iota)\Delta(a) = h(a)1 \text{ for all } a\in C(G).$$

We write $L^2(G)$ for the Hilbert space of the GNS-representation defined by h.

Example 1.2.2. — If $G=\hat{\Gamma}$ for a discrete group Γ, so $C(G)=C_r^*(\Gamma)$, then the Haar state is the canonical trace on $C_r^*(\Gamma)$. The same is true if we take $C(G)=C^*(\Gamma)$. This shows that in general the Haar state is not faithful. □

References. — [**62**], [**94**], [**97**].

1.3. REPRESENTATION THEORY

We want to define the notion of a representation of a compact quantum group on a finite dimensional vector space. For this we first have to introduce some notation.

Given a unital algebra $\mathscr{A}$, natural numbers $n\le m$ and an injective map $\sigma\colon\{1,\dots,n\}\to\{1,\dots,m\}$, we can define an obvious embedding $\mathscr{A}^{\otimes n}\hookrightarrow\mathscr{A}^{\otimes m}$. The image of an element $X\in\mathscr{A}^{\otimes n}$ under this embedding is denoted by $X_{j_1\dots j_n}$, where $j_i=\sigma(i)$. For example, if $n=2$ and $m=4$, then

$$(a\otimes b)_{31} = b\otimes 1\otimes a\otimes 1.$$

This can be slightly generalized by considering embeddings of tensor products of different unital algebras of the form $\mathscr{A}_1\otimes\dots\otimes\mathscr{A}_n\hookrightarrow\mathscr{B}_1\otimes\dots\otimes\mathscr{B}_m$, with $\mathscr{B}_{\sigma(i)}=\mathscr{A}_i$. What is taken for the number m and for the algebras $\mathscr{B}_j$ for $j\notin\operatorname{Im}\sigma$ is usually clear from the context. This is called the leg-numbering notation.

Turning to representations, consider first a genuine compact group G. Recall that a representation of G on a finite dimensional space H is a continuous homomorphism $G\to\mathrm{GL}(H)$, $g\mapsto U_g$. Identifying the algebra $C(G;B(H))$ of continuous $B(H)$-valued functions on G with $B(H)\otimes C(G)$, we can consider the function $g\mapsto U_g$ as an element $U\in B(H)\otimes C(G)$. Then the condition $U_sU_t=U_{st}$ for all $s,t\in G$ can be equivalently written as $(\iota\otimes\Delta)(U)=U_{12}U_{13}$. This motivates the following definition.

Definition 1.3.1. — A representation of a compact quantum group G on a finite dimensional vector space H is an invertible element U of $B(H)\otimes C(G)$ such that

$$(\iota\otimes\Delta)(U) = U_{12}U_{13} \text{ in } B(H)\otimes C(G)\otimes C(G).$$

The representation is called unitary if H is a Hilbert space and U is unitary.

We will often denote a representation by one symbol U and write H_U for the underlying space.

If $\xi_1,\dots,\xi_n$ is a basis in H, m_{ij} the corresponding matrix units in $B(H)$, so $m_{ij}\xi_k = \delta_{jk}\xi_i$, then the condition $(\iota\otimes\Delta)(U) = U_{12}U_{13}$ for $U = \sum_{i,j} m_{ij}\otimes u_{ij} \in B(H)\otimes C(G)$ reads as

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik}\otimes u_{kj}.$$

In particular, the unitaries $U = (u_{ij})_{i,j}$ introduced in Examples 1.1.5-1.1.8 define unitary representations of the corresponding quantum groups. These representations are called the fundamental representations.

We can obviously take direct sums of representations. We can also define tensor products.

Definition 1.3.2. — The tensor product of two finite dimensional representations U and V is the representation $U\times V$ on $H_U\otimes H_V$ defined by $U\times V = U_{13}V_{23}$.

Another commonly used notation for the tensor product is $U \mathbin{\bigcirc\!\!\!\!\!\top} V$.

Definition 1.3.3. — Assume U and V are finite dimensional representations. We say that an operator $T\colon H_U\to H_V$ intertwines U and V if

$$(T\otimes 1)U = V(T\otimes 1).$$

Denote by $\mathrm{Mor}(U,V)$ the space of intertwiners. Representations U and V are called equivalent if $\mathrm{Mor}(U,V)$ contains an invertible element. Correspondingly, unitary representations U and V are called unitarily equivalent if $\mathrm{Mor}(U,V)$ contains a unitary element.

We write $\mathrm{End}(U)$ for $\mathrm{Mor}(U,U)$. A representation U is called irreducible if $\mathrm{End}(U) = \mathbb{C}$.

If U is a unitary representation, then $\mathrm{End}(U)$ is a C*-algebra. More generally, if $T\in \mathrm{Mor}(U,V)$ and U and V are unitary, then $T^*\in\mathrm{Mor}(V,U)$. This allows us to prove the following simple, but fundamental, result.

***Proposition 1.3.4* (Schur's lemma)**. — *Two irreducible unitary representations U and V are either unitarily equivalent and* $\mathrm{Mor}(U,V)$ *is one-dimensional, or* $\mathrm{Mor}(U,V) = 0$.

Proof. — Assume $T\colon H_U\to H_V$ is a nonzero intertwiner. Then $T^*T\in\mathrm{End}(U)$ and $TT^*\in\mathrm{End}(V)$ are nonzero scalars. It follows that up to a scalar factor the operator T is unitary. If $S\colon H_U\to H_V$ is another intertwiner, then $T^*S\in\mathrm{End}(U)$ is a scalar operator, hence S is a scalar multiple of T. Therefore $\mathrm{Mor}(U,V) = \mathbb{C}T$. □

The study of finite dimensional representations can be reduced to that of unitary ones.

Proposition 1.3.5. — *Every finite dimensional representation is equivalent to a unitary representation.*

Proof. — Let U be a finite dimensional representation. Take any Hilbert space structure on H_U. Consider the operator

$$Q = (\iota \otimes h)(U^*U) \in B(H_U).$$

Since U is invertible, $U^*U \geq \varepsilon 1$ for some $\varepsilon > 0$, hence $Q \geq \varepsilon 1$.

We have

$$(\iota \otimes \Delta)(U^*U) = U_{13}^*U_{12}^*U_{12}U_{13}.$$

Applying $\iota \otimes h \otimes \iota$ and using that $(h \otimes \iota)\Delta(\cdot) = h(\cdot)1$, we get

$$Q \otimes 1 = U^*(Q \otimes 1)U.$$

Therefore

$$V = (Q^{1/2} \otimes 1)U(Q^{-1/2} \otimes 1)$$

is a unitary representation of G on H_U, and $Q^{1/2} \in \mathrm{Mor}(U, V)$. □

Remark 1.3.6. — The same computation as in the above proof gives the following result, which we will need several times later. Suppose we have elements $U \in B(H_U) \otimes C(G)$ and $V \in B(H_V) \otimes C(G)$ such that $(\iota \otimes \Delta)(U) = U_{12}U_{13}$ and $(\iota \otimes \Delta)(V) = V_{12}V_{13}$. Let $S\colon H_U \to H_V$ be a linear operator. Define

$$T = (\iota \otimes h)(V^*(S \otimes 1)U) \in B(H_U, H_V).$$

Then $V^*(T \otimes 1)U = T \otimes 1$. In particular, if U and V are unitary, then $T \in \mathrm{Mor}(U, V)$.

Similarly, if we put $T = (\iota \otimes h)(V(S \otimes 1)U^*)$, then $V(T \otimes 1)U^* = T \otimes 1$.

Theorem 1.3.7. — *Every finite dimensional representation is a direct sum of irreducible representations.*

Proof. — Let U be a finite dimensional representation. We may assume that U is unitary. Then $\mathrm{End}(U)$ is a C*-algebra. Let $e_1, \dots, e_n$ be minimal projections in $\mathrm{End}(U)$ that add up to 1. Then $(e_i \otimes 1)U$ are irreducible representations of G on e_iH, and their direct sum is U. □

Next we want to define contragredient representations. If G is a group and $g \mapsto U_g$ is a representation of G on H, then the contragredient representation is the representation U^c on the dual space H^* defined by $(U_g^c f)(\xi) = f(U_g^{-1}\xi)$ for $f \in H^*$ and $\xi \in H$. When H is a Hilbert space, we identify H^* with the complex conjugate Hilbert space $\bar{H}$. In this case, if U is unitary, then U^c is also unitary. As we will see, the analogous property for quantum groups is, in general, not true.

Assume now that G is a compact quantum group and $U \in B(H) \otimes C(G)$ is a finite dimensional representation. Consider the dual space H^* and denote by $j\colon B(H) \to$

$B(H^*)$ the map that sends an operator to the dual operator. We will use the same symbol j for all spaces, so that in particular j^2 is well-defined and equals the identity map. If H is a Hilbert space, so that $H^* = \bar{H}$, then $j(T)\bar{\xi} = \overline{T^*\xi}$. Note that in this case j is a $*$-anti-homomorphism.

Definition 1.3.8. — The contragredient representation to a representation U is the representation U^c on the dual space defined by

$$U^c = (j \otimes \iota)(U^{-1}) \in B(H^*) \otimes C(G).$$

It is by no means obvious that U^c is indeed a representation: the problem is to show that the element U^c is invertible.

Note that if U is unitary and U is written as a matrix $(u_{ij})_{i,j}$ with respect to an orthonormal basis in H, then $U^c = (u^*_{ij})_{i,j}$ with respect to the dual basis in $\bar{H}$. In particular, we can now fully understand where the assumptions of Proposition 1.1.4 come from.

Returning to general U, observe that as $(\iota \otimes \Delta)(U^{-1}) = U^{-1}_{13}U^{-1}_{12}$ and j is an anti-homomorphism, we have $(\iota \otimes \Delta)(U^c) = U^c_{12}U^c_{13}$. But in order to show that U^c is invertible we need some preparation.

Proposition 1.3.9. — *Assume $U \in B(H) \otimes C(G)$ is a finite dimensional unitary representation. Consider the space*

$$B = \{(\iota \otimes h)(U(1 \otimes a)) \mid a \in C(G)\} \subset B(H).$$

Then B is a $*$-subalgebra of $B(H)$ containing the unit of $B(H)$, and $U \in B \otimes C(G)$.

Proof. — For $a \in C(G)$ put $L(a) = (\iota \otimes h)(U(1 \otimes a)) \in B(H)$. We have

$$U^*_{12}(\iota \otimes \Delta)(U(1 \otimes a)) = U_{13}(1 \otimes \Delta(a)).$$

Applying $\iota \otimes \iota \otimes h$ we get

$$U^*(L(a) \otimes 1) = (\iota \otimes \iota \otimes h)(U_{13}(1 \otimes \Delta(a))). \tag{1.3.1}$$

Since $L(b)^* = (\iota \otimes h)((1 \otimes b^*)U^*)$, multiplying the above identity by $1 \otimes b^*$ and applying $\iota \otimes h$ we get

$$L(b)^*L(a) = (\iota \otimes h \otimes h)(U_{13}(1 \otimes (b^* \otimes 1)\Delta(a))).$$

This shows that $L(b)^*L(a) \in B$ for any $a, b \in C(G)$ and furthermore, since the space $(C(G) \otimes 1)\Delta(C(G))$ is dense in $C(G) \otimes C(G)$, that B is spanned by elements of this form. Hence B is self-adjoint, so it consists of the elements $L(b)^*$, and therefore B is a $*$-algebra.

Since $U(B(H) \otimes C(G)) = B(H) \otimes C(G)$, we have $B \cdot B(H) = B(H)$. Therefore the representation of B on H is nondegenerate. Hence B contains the identity operator.

From (1.3.1) we see then that U^* belongs to $B \otimes C(G)$, so $U \in B \otimes C(G)$. ☐

As $U \in B \otimes C(G)$, by definition of B we conclude that $\mathrm{End}(U)$ is the commutant of B in $B(H)$. It follows that U is irreducible if and only if $B = B(H)$.

Corollary 1.3.10. — *Assume $U \in B(H) \otimes C(G)$ is an irreducible finite dimensional unitary representation. Then $(X \otimes 1)U(Y \otimes 1) \neq 0$ for any nonzero elements $X, Y \in B(H)$.*

Proof. — If $(X \otimes 1)U(Y \otimes 1) = 0$, then $XBY = 0$. Since $B = B(H)$, this is possible only when $X = 0$ or $Y = 0$. $\square$

We are now ready to prove that U^c is indeed a representation.

Proposition 1.3.11. — *For any representation U on a finite dimensional space H the element $U^c \in B(H^*) \otimes C(G)$ is invertible.*

Proof. — Unitarizing and decomposing the representation into irreducibles, we may assume that U is unitary and irreducible. Consider the positive operators

$$Q_\ell = (\iota \otimes h)(U^c U^{c*}) \ \text{ and } \ Q_r = (\iota \otimes h)(U^{c*} U^c)$$

in $B(\bar{H})$. By Remark 1.3.6 we have

$$Q_\ell \otimes 1 = U^c (Q_\ell \otimes 1) U^{c*} \ \text{ and } \ Q_r \otimes 1 = U^{c*}(Q_r \otimes 1) U^c. \tag{1.3.2}$$

Therefore in order to prove that U^c is invertible it suffices to show that Q_ℓ and Q_r are invertible.

Let us show first that $Q_\ell \neq 0$. For this we compute the trace of Q_ℓ:

$$\begin{aligned}\mathrm{Tr}(Q_\ell) &= (\mathrm{Tr} \otimes h)(U^c U^{c*}) = (\mathrm{Tr} \otimes h)((j \otimes \iota)(U^*)(j \otimes \iota)(U)) \\ &= (\mathrm{Tr} \otimes h)(U^* U) = \dim H,\end{aligned}$$

where we used that $\mathrm{Tr}(j(X)j(Y)) = \mathrm{Tr}(j(YX)) = \mathrm{Tr}(YX) = \mathrm{Tr}(XY)$.

Let $p \in B(\bar{H})$ be the projection onto the kernel of Q_ℓ. From (1.3.2) we get

$$(p \otimes 1)U^c (Q_\ell \otimes 1) U^{c*}(p \otimes 1) = 0,$$

whence $(Q_\ell^{1/2} \otimes 1)U^{c*}(p \otimes 1) = 0$, or in other words,

$$(Q_\ell^{1/2} \otimes 1)(j \otimes \iota)(U)(p \otimes 1) = 0.$$

Hence $(j(p) \otimes 1)U(j(Q_\ell^{1/2}) \otimes 1) = 0$. Since $Q_\ell \neq 0$, by Corollary 1.3.10 this is possible only when $p = 0$. Therefore Q_ℓ is invertible. Similarly one proves that Q_r is invertible. $\square$

Let $U \in B(H) \otimes C(G)$ be an irreducible finite dimensional unitary representation. In the proof of the previous proposition we introduced positive invertible operators $Q_\ell, Q_r \in B(\bar{H})$ by

$$Q_\ell = (\iota \otimes h)(U^c U^{c*}) \ \text{ and } \ Q_r = (\iota \otimes h)(U^{c*} U^c).$$

We shall now investigate their properties.

Lemma 1.3.12. — *We have:*

(i) $j(Q_\ell) \in \mathrm{Mor}(U^{cc}, U)$ *and* $j(Q_r) \in \mathrm{Mor}(U, U^{cc})$*;*

(ii) $Q_\ell Q_r$ *is a scalar;*

(iii) $\mathrm{Tr}(Q_\ell) = \mathrm{Tr}(Q_r) = \dim H_U$.

Proof. — (i) As we have already used in the proof of Proposition 1.3.11, by Remark 1.3.6 we have

$$U^c(Q_\ell \otimes 1)U^{c*} = Q_\ell \otimes 1 \ \text{ and } \ U^{c*}(Q_r \otimes 1)U^c = Q_r \otimes 1.$$

Multiplying the first equality by $(U^c)^{-1}$ on the left we get

$$(Q_\ell \otimes 1)(j \otimes \iota)(U) = (U^c)^{-1}(Q_\ell \otimes 1).$$

Therefore applying $j \otimes \iota$ we obtain

$$U(j(Q_\ell) \otimes 1) = (j(Q_\ell) \otimes 1)U^{cc}.$$

Thus $j(Q_\ell) \in \mathrm{Mor}(U^{cc}, U)$. Similarly one shows that $j(Q_r) \in \mathrm{Mor}(U, U^{cc})$.

(ii) Since U is irreducible and $j(Q_\ell)j(Q_r) \in \mathrm{End}(U)$ by (i), the operator $j(Q_\ell)j(Q_r)$ is a scalar.

(iii) That $\mathrm{Tr}(Q_\ell) = \dim H_U$ was shown in the proof of Proposition 1.3.11. The equality $\mathrm{Tr}(Q_r) = \dim H_U$ is proved similarly. □

Remark 1.3.13. — The proof of (i) shows that for any finite dimensional unitary representation U and an operator $Q \in B(H_U)$, we have $Q \in \mathrm{Mor}(U^{cc}, U)$ if and only if $U^c(j(Q) \otimes 1)U^{c*} = j(Q) \otimes 1$. Similarly, $Q \in \mathrm{Mor}(U, U^{cc})$ if and only if $U^{c*}(j(Q) \otimes 1)U^c = j(Q) \otimes 1$.

Let us now list properties of the contragredient representation.

Proposition 1.3.14. — *For any finite dimensional representations we have:*

(i) U^{cc} *is equivalent to* U*;*

(ii) U^c *is irreducible if and only if* U *is irreducible;*

(iii) *the flip map* $H_U^* \otimes H_V^* \to H_V^* \otimes H_U^*$ *defines an equivalence between* $(U \times V)^c$ *and* $V^c \times U^c$.

Proof. — Part (i) for irreducible representations follows from Lemma 1.3.12(i). By decomposing representations into irreducible, we get the result in general. Part (ii) is true since $T \in \mathrm{End}(U)$ if and only if $j(T) \in \mathrm{End}(U^c)$. Part (iii) follows immediately from our definitions. □

References. — [**62**], [**94**], [**97**].

1.4. QUANTUM DIMENSION AND ORTHOGONALITY RELATIONS

Let U be an irreducible finite dimensional unitary representation of a compact quantum group G. By irreducibility and Lemma 1.3.12(i) the space $\mathrm{Mor}(U, U^{cc})$ is one-dimensional, spanned by a unique up to a scalar factor positive invertible operator. Denote by $\rho_U \in B(H_U)$ this unique operator normalized so that $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$.

Definition 1.4.1. — The quantum dimension of U is the number

$$\dim_q U = \mathrm{Tr}(\rho_U).$$

Since $\dim H_U \le \mathrm{Tr}(X)^{1/2}\mathrm{Tr}(X^{-1})^{1/2}$ for any $X > 0$ by the Cauchy-Schwarz inequality, we have $\dim_q U \ge \dim H_U$, and equality holds if and only if $\rho_U = 1$. We will write $\dim U$ for $\dim H_U$. As follows from results of the previous section, and will be discussed shortly, we have $\rho_U = 1$ if and only if U^c is unitary. Therefore the ratio $\dim_q U/\dim U \ge 1$ in some sense measures how far the contragredient representation is from being unitary.

Example 1.4.2. — Let G be any of the quantum groups introduced in Examples 1.1.5-1.1.7, and $U = (u_{ij})_{i,j}$ be its fundamental representation. It can be shown that this representation is irreducible (we will prove this later for $SU_q(2)$ and $A_o(F)$). In all these examples we are given a matrix F such that FU^cF^{-1} is unitary and $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$. Therefore $(F^{-1})^*U^{c*}F^*FU^cF^{-1} = 1$, so by Remark 1.3.13 we have $j(F^*F) = (F^*F)^t \in \mathrm{Mor}(U, U^{cc})$. Thus

$$\rho_U = (F^*F)^t.$$

In particular, for $G = SU_q(2)$ we have $F = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$, with $\lambda^2 = q$. Hence

$$\rho_U = \begin{pmatrix} |q|^{-1} & 0 \\ 0 & |q| \end{pmatrix}$$

and $\dim_q U = |q + q^{-1}|$. □

We extend the quantum dimension to all finite dimensional representations by additivity. The name 'dimension' is justified by the properties that we are now going to establish, see also Section 2.2.

Somewhat more explicitly $\dim_q U$ for irreducible unitary representations is defined as follows. Consider the operators $Q_\ell, Q_r \in B(\bar{H}_U)$ defined by

$$Q_\ell = (\iota \otimes h)(U^cU^{c*}) \text{ and } Q_r = (\iota \otimes h)(U^{c*}U^c).$$

By Lemma 1.3.12, $j(Q_r) \in \mathrm{Mor}(U, U^{cc})$, $Q_\ell Q_r$ is a scalar operator $\lambda 1$ and $\mathrm{Tr}(Q_\ell) = \mathrm{Tr}(Q_r) = \dim U$. It follows that

$$\rho_U = \lambda^{-1/2} j(Q_r), \quad \rho_U^{-1} = \lambda^{-1/2} j(Q_\ell), \quad \dim_q U = \lambda^{-1/2} \dim U.$$

Note that, conversely, Q_r and Q_ℓ are expressed in terms of ρ_U by

$$Q_r = \frac{\dim U}{\dim_q U} j(\rho_U) \quad \text{and} \quad Q_\ell = \frac{\dim U}{\dim_q U} j(\rho_U^{-1}). \tag{1.4.1}$$

***Theorem 1.4.3* (Orthogonality relations).** — *Let $U = (u_{ij})_{i,j}$ be an irreducible unitary representation written in matrix form with respect to an orthonormal basis in H_U, and let $\rho = \rho_U$. Then*

(i) $h(u_{kl} u_{ij}^*) = \dfrac{\delta_{ki} \rho_{jl}}{\dim_q U}$ *and* $h(u_{ij}^* u_{kl}) = \dfrac{\delta_{jl} (\rho^{-1})_{ki}}{\dim_q U}$;

(ii) *if $V = (v_{kl})_{k,l}$ is an irreducible unitary representation that is not equivalent to U, then* $h(v_{kl} u_{ij}^*) = h(u_{ij}^* v_{kl}) = 0$.

Proof. — (i) For any $T \in B(H_U)$, by Remark 1.3.6 we have $(\iota \otimes h)(U(T \otimes 1)U^*) \in \mathrm{End}(U) = \mathbb{C}1$. Therefore there exists a unique positive operator $\rho_r \in B(H_U)$ such that

$$\mathrm{Tr}(\rho_r T) 1 = (\iota \otimes h)(U(T \otimes 1)U^*) \quad \text{for all } T \in B(H_U). \tag{1.4.2}$$

Taking the trace and using that $\mathrm{Tr}(XYZ) = \mathrm{Tr}(YZX) = \mathrm{Tr}(j(X)j(Z)j(Y))$, we get that

$$\begin{aligned}(\dim U)\,\mathrm{Tr}(\rho_r T) &= (\mathrm{Tr} \otimes h)(U(T \otimes 1)U^*)\\ &= (\mathrm{Tr} \otimes h)((j \otimes \iota)(U)(j \otimes \iota)(U^*)(j(T) \otimes 1))\\ &= (\mathrm{Tr} \otimes h)(U^{c*} U^c (j(T) \otimes 1))\\ &= \mathrm{Tr}(Q_r j(T)) = \mathrm{Tr}(j(Q_r) T).\end{aligned}$$

Therefore, by (1.4.1), we have

$$\rho_r = \frac{j(Q_r)}{\dim U} = \frac{\rho}{\dim_q U}. \tag{1.4.3}$$

As one can now easily check, identity (1.4.2) applied to $T = m_{lj}$ gives the first equality in (i). The second equality is proved similarly, by showing that

$$\frac{\mathrm{Tr}(\rho^{-1} T)}{\dim_q U} 1 = (\iota \otimes h)(U^*(T \otimes 1)U) \quad \text{for all } T \in B(H_U).$$

(ii) Using again Remark 1.3.6 and that $\mathrm{Mor}(V, U) = 0$ and $\mathrm{Mor}(U, V) = 0$, we get that

$$(\iota \otimes h)(V(S \otimes 1)U^*) = 0 \quad \text{and} \quad (\iota \otimes h)(U^*(T \otimes 1)V) = 0$$

for all $S\colon H_U \to H_V$ and $T\colon H_V \to H_U$. This is equivalent to the identities in (ii). □

In order to study properties of the quantum dimension we need to define the operators ρ_U for all unitary representations.

Proposition 1.4.4. — *Let U be a finite dimensional unitary representation. Then there exists a unique positive invertible operator $\rho \in \mathrm{Mor}(U, U^{cc})$ such that*

$$\mathrm{Tr}(\cdot\,\rho) = \mathrm{Tr}(\cdot\,\rho^{-1}) \ \ \textit{on} \ \ \mathrm{End}(U) \subset B(H_U).$$

Proof. — There exist pairwise nonequivalent irreducible unitary representations U_i, $1 \le i \le n$, such that U decomposes into a direct sum of copies of U_i. In other words, we may assume that $H_U = \oplus_i (K_i \otimes H_{U_i})$ for some finite dimensional Hilbert spaces K_i, and $U = \oplus_i (1 \otimes U_i)$. In this case $\mathrm{End}(U) = \oplus_i (B(H_i) \otimes 1)$. From this we see that the operator $\rho = \oplus_i (1 \otimes \rho_{U_i})$ has the required properties.

Assume $\rho' \in B(H_U)$ is another positive invertible operator with the same properties. Then $\rho^{-1}\rho' \in \mathrm{End}(U)$, so $\rho' = \oplus_i (T_i \otimes \rho_{U_i})$ for some positive invertible operators $T_i \in B(K_i)$. Then $\mathrm{Tr}(\cdot\, T_i) = \mathrm{Tr}(\cdot\, T_i^{-1})$ on $B(K_i)$. Hence $T_i = T_i^{-1}$, and so $T_i = 1$, as T_i are positive. Therefore $\rho' = \rho$. □

We denote by $\rho_U \in B(H_U)$ the unique operator given by the above proposition. By definition we have $\dim_q U = \mathrm{Tr}(\rho_U)$.

As follows from Remark 1.3.13, a positive invertible operators $\rho \in B(H_U)$ belongs to $\mathrm{Mor}(U, U^{cc})$ if and only if the operator

$$(j(\rho)^{1/2} \otimes 1)U^c(j(\rho)^{-1/2} \otimes 1)$$

is unitary.

Definition 1.4.5. — For a finite dimensional unitary representation U the conjugate representation is defined by

$$\bar{U} = (j(\rho_U)^{1/2} \otimes 1)U^c(j(\rho_U)^{-1/2} \otimes 1) \in B(\bar{H}_U) \otimes C(G).$$

The conjugate representation is therefore a canonical unitary form of the contragredient representation.

Note that the contragredient representation U^c is unitary if and only if $\rho_U = 1$. Indeed, if $\rho_U = 1$, then $U^c = \bar{U}$, so U^c is unitary. Conversely, if U^c is unitary, then V^c is unitary for any irreducible unitary subrepresentation V of U. Since by construction ρ_V is a scalar multiple of $(j \otimes h)(V^{c*}V^c)$, it follows that $\rho_V = 1$. Since this is true for all subrepresentations V, we get $\rho_U = 1$ by construction of ρ_U.

Note also that if $\rho \in \mathrm{Mor}(U, U^{cc})$ is positive and invertible, then by the proof of Proposition 1.4.4 we have $\rho = \rho_U T$ for some $T \in \mathrm{End}(U)$ commuting with ρ_U. Hence

$$\bar{U} = (j(\rho)^{1/2} \otimes 1)U^c(j(\rho)^{-1/2} \otimes 1).$$

Using this and the property $\rho_{U\oplus V} = \rho_U \oplus \rho_V$, which follows by construction, it is easy to check the following properties of the conjugate representation.

Proposition 1.4.6. — *For any finite dimensional unitary representations we have:*
(i) $\bar{\bar{U}} = U$;
(ii) $\overline{U \oplus V} = \bar{U} \oplus \bar{V}$;
(iii) *the flip* $\bar{H}_U \otimes \bar{H}_V \to \bar{H}_V \otimes \bar{H}_U$ *defines an equivalence between* $\overline{U \times V}$ *and* $\bar{V} \times \bar{U}$.

In order to compute the quantum dimensions of $\bar{U}$ and $U \times V$ we, however, need to find $\rho_{\bar{U}}$ and $\rho_{U\times V}$ explicitly.

Proposition 1.4.7. — *For any finite dimensional unitary representation U we have $\rho_{\bar{U}} = j(\rho_U)^{-1}$. In particular,* $\dim_q \bar{U} = \dim_q U$.

Proof. — Since $\rho_{V_1 \oplus V_2} = \rho_{V_1} \oplus \rho_{V_2}$, it suffices to consider irreducible representations. We have

$$
\begin{aligned}
(\bar{U})^c &= (j \otimes \iota)(\bar{U})^* = (j \otimes \iota)((j(\rho_U)^{-1/2} \otimes 1)U^{c*}(j(\rho_U)^{1/2} \otimes 1)) \\
&= (\rho_U^{1/2} \otimes 1)(j \otimes \iota)(U^{c*})(\rho_U^{-1/2} \otimes 1) = (\rho_U^{1/2} \otimes 1)U(\rho_U^{-1/2} \otimes 1).
\end{aligned}
$$

Thus $(\rho_U^{-1/2} \otimes 1)(\bar{U})^c(\rho_U^{1/2} \otimes 1)$ is unitary. As we remarked after the proof of Proposition 1.4.4, this is equivalent to $j(\rho_U)^{-1} \in \mathrm{Mor}(\bar{U}, (\bar{U})^{cc})$. Since $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$, we conclude that $\rho_{\bar{U}} = j(\rho_U)^{-1}$. □

In order to deal with tensor products we introduce auxiliary states on $B(H_U)$ by

$$\varphi_U = \frac{\mathrm{Tr}(\cdot\, \rho_U^{-1})}{\dim_q U}, \quad \psi_U = \frac{\mathrm{Tr}(\cdot\, \rho_U)}{\dim_q U}.$$

By definition of ρ_U we have $\varphi_U = \psi_U$ on $\mathrm{End}(U)$.

Lemma 1.4.8. — *For any finite dimensional unitary representations U and V we have*

$$(\varphi_U \otimes \iota)(\mathrm{End}(U \times V)) \subset \mathrm{End}(V) \quad \textit{and} \quad (\iota \otimes \psi_V)(\mathrm{End}(U \times V)) \subset \mathrm{End}(U).$$

Proof. — We will only prove the second inclusion, the first one is proved similarly. We claim that

$$(\psi_V \otimes \iota)(V(S \otimes 1)V^*) = \psi_V(S)1 \ \text{ for any } S \in B(H_V). \tag{1.4.4}$$

It suffices to show this assuming that V is irreducible. In this case, by (1.4.2) and (1.4.3), we have that

$$(\iota \otimes h)(V(S \otimes 1)V^*) = \psi_V(S)1.$$

Hence

$$
\begin{aligned}
(\psi_V \otimes \iota)(V(S \otimes 1)V^*) &= (\psi_V \otimes h \otimes \iota)(V_{12}V_{13}(S \otimes 1 \otimes 1)V_{13}^*V_{12}^*) \\
&= (\psi_V \otimes h \otimes \iota)((\iota \otimes \Delta)(V)(S \otimes 1 \otimes 1)(\iota \otimes \Delta)(V^*)) \\
&= (\psi_V \otimes h)(V(S \otimes 1)V^*)1 = \psi_V(S)1,
\end{aligned}
$$

and (1.4.4) is proved.

Therefore, if $T \in \mathrm{End}(U \times V)$, then

$$\begin{aligned}
U((\iota\otimes\psi_V)(T)\otimes 1)U^* &= (\iota\otimes\psi_V\otimes\iota)(U_{13}(T\otimes 1)U_{13}^*)\\
&= (\iota\otimes\psi_V\otimes\iota)(U_{13}V_{23}(T\otimes 1)V_{23}^*U_{13}^*)\\
&= (\iota\otimes\psi_V\otimes\iota)((U\times V)(T\otimes 1)(U\times V)^*)\\
&= (\iota\otimes\psi_V\otimes\iota)(T\otimes 1) = (\iota\otimes\psi_V)(T)\otimes 1,
\end{aligned}$$

so $(\iota\otimes\psi_V)(T) \in \mathrm{End}(U)$. □

Theorem 1.4.9. — *For any finite dimensional unitary representations U and V we have $\rho_{U\times V} = \rho_U \otimes \rho_V$. In particular,* $\dim_q(U\times V) = \dim_q U \cdot \dim_q V$.

Proof. — We will use the criterion from Proposition 1.4.4. The only nontrivial property to check is that $\mathrm{Tr}(\cdot\,\rho_U\otimes\rho_V) = \mathrm{Tr}(\cdot\,\rho_U^{-1}\otimes\rho_V^{-1})$ on $\mathrm{End}(U\times V)$, that is,

$$\psi_U\otimes\psi_V = \varphi_U\otimes\varphi_V \quad \text{on} \quad \mathrm{End}(U\times V).$$

If $T \in \mathrm{End}(U\times V)$, then using Lemma 1.4.8 and that $\psi_U = \varphi_U$ on $\mathrm{End}(U)$, we get

$$(\psi_U\otimes\psi_V)(T) = \psi_U(\iota\otimes\psi_V)(T) = \varphi_U(\iota\otimes\psi_V)(T) = (\varphi_U\otimes\psi_V)(T).$$

Similarly, $(\varphi_U\otimes\varphi_V)(T) = \varphi_V(\varphi_U\otimes\iota)(T) = (\varphi_U\otimes\psi_V)(T)$. □

References. — [**59**], [**94**], [**97**].

1.5. INFINITE DIMENSIONAL REPRESENTATIONS

So far we have dealt only with finite dimensional representations. However, the only known way to show that there exist many such representations is by decomposing infinite dimensional representations.

For a Hilbert space H we denote by $K(H)$ the C*-algebra of compact operators on H. Recall also that for a C*-algebra A the multiplier algebra of A is denoted by $M(A)$. If $A \subset B(H)$ and AH is dense in H, then $M(A)$ can be identified with the C*-subalgebra of $B(H)$ consisting of the operators $T \in B(H)$ such that $Ta \in A$ and $aT \in A$ for all $a \in A$.

Definition 1.5.1. — A unitary representation of a compact quantum group G on a Hilbert space H is a unitary element $U \in M(K(H)\otimes C(G))$ such that

$$(\iota\otimes\Delta)(U) = U_{12}U_{13}.$$

Our goal is to construct a representation of G on $L^2(G)$ that coincides with the right regular representation in the group case. Recall that $L^2(G)$ denotes the underlying space of the GNS-representation π_h of $C(G)$ defined by the Haar state h. Write $\Lambda(a)$

for $a \in C(G)$ considered as a vector in $L^2(G)$. We will often suppress π_h in the computations. Therefore $a\Lambda(b) = \pi_h(a)\Lambda(b) = \Lambda(ab)$.

Let us first consider the group case. The right regular representation $g \mapsto W_g$ of a compact group G is defined by $(W_s f)(t) = f(ts)$ for $f \in L^2(G)$. The multiplier algebra $M(K(L^2(G)) \otimes C(G))$ can be identified with the algebra of strongly* operator continuous functions $G \to B(L^2(G))$. Therefore the operators W_g define a unitary $W \in M(K(L^2(G)) \otimes C(G))$, so we indeed have a representation of G in the sense of the above definition. Note that for $f_1 \in C(G)$, $f_2 \in L^2(G)$ and $s, t \in G$ we have

$$(W(f_1 \otimes f_2))(s,t) = (W_t f_1)(s) f_2(t) = f_1(st) f_2(t) = (\Delta(f_1)(1 \otimes f_2))(s,t).$$

This motivates the following construction.

Theorem 1.5.2. — *For any compact quantum group G, there exists a unique unitary representation W of G on the space $L^2(G)$ such that if $C(G) \subset B(H_0)$, then*

$$W(\Lambda(a) \otimes \xi) = \Delta(a)(\Lambda(1) \otimes \xi) \ \ \textit{for all } a \in C(G) \ \textit{ and } \ \xi \in H_0.$$

Proof. — We divide the proof into several steps. Assume $C(G) \subset B(H_0)$. Let us also assume that the representation of $C(G)$ on H_0 is universal, in the sense that any bounded linear functional ω on $C(G)$ has the form $\omega = \omega_{\xi,\zeta}$ for some $\xi, \zeta \in H_0$, where $\omega_{\xi,\zeta}(T) = (T\xi, \zeta)$.

Step 1. The equality in the formulation defines an isometry W on $L^2(G) \otimes H_0$.

Write ξ_h for $\Lambda(1)$. For $a_i \in C(G)$ and $\xi_i \in H_0$, $1 \le i \le n$, we have

$$\begin{aligned} \Big\| \sum_i \Delta(a_i)(\xi_h \otimes \xi_i) \Big\|^2 &= \sum_{i,j} (\Delta(a_j^* a_i)(\xi_h \otimes \xi_i), \xi_h \otimes \xi_j) \\ &= \sum_{i,j} ((h \otimes \iota)\Delta(a_j^* a_i)\xi_i, \xi_j) \\ &= \sum_{i,j} (h(a_j^* a_i)\xi_i, \xi_j) \\ &= \Big\| \sum_i \Lambda(a_i) \otimes \xi_i \Big\|^2 . \end{aligned}$$

Hence W is a well-defined isometry.

Step 2. The operator W is unitary.

Indeed, the image of W contains all vectors of the form $\Delta(a)(1 \otimes b)(\xi_h \otimes \xi)$, so it is dense by the cancellation property.

Step 3. We have $W \in M(K(L^2(G)) \otimes C(G))$.

We have to show that $W(T\otimes 1), W^*(T\otimes 1) \in K(L^2(G))\otimes C(G)$ for any $T \in K(L^2(G))$. It suffices to consider $T = \theta_{a\xi_h,\xi}$, where $\theta_{\eta,\xi}\zeta = (\zeta,\xi)\eta$. We have

$$W(\theta_{a\xi_h,\xi}\otimes 1)(\zeta\otimes\eta) = W((\zeta,\xi)a\xi_h\otimes\eta) = (\zeta,\xi)\Delta(a)(\xi_h\otimes\eta)$$
$$= \Delta(a)(\theta_{\xi_h,\xi}\otimes 1)(\zeta\otimes\eta).$$

If $\sum_i a_i\otimes b_i$ is close to $\Delta(a)$, we see that $W(\theta_{a\xi_h,\xi}\otimes 1)$ is close to $\sum\limits_i \theta_{a_i\xi_h,\xi}\otimes b_i$. Thus $W(T\otimes 1)\in K(L^2(G))\otimes C(G)$.

Turning to $W^*(T\otimes 1)$, assume that $\sum_i \Delta(a_i)(1\otimes b_i)$ is close to $a\otimes 1$. Then the identity

$$\sum_i W^*\Delta(a_i)(1\otimes b_i)(\theta_{\xi_h,\xi}\otimes 1)(\zeta\otimes\eta) = (\zeta,\xi)\sum_i W^*\Delta(a_i)(\xi_h\otimes b_i\eta)$$
$$= (\zeta,\xi)\sum_i a_i\xi_h\otimes b_i\eta = \left(\sum_i \theta_{a_i\xi_h,\xi}\otimes b_i\right)(\zeta\otimes\eta)$$

shows that $W^*(\theta_{a\xi_h,\xi}\otimes 1) = W^*(a\otimes 1)(\theta_{\xi_h,\xi}\otimes 1)$ is close in norm to $\sum_i \theta_{a_i\xi_h,\xi}\otimes b_i$. Hence $W^*(T\otimes 1)\in K(L^2(G))\otimes C(G)$.

Step 4. We have $(\iota\otimes\Delta)(W) = W_{12}W_{13}$.

For $a\in C(G)$ and $\omega\in C(G)^*$ define

$$\omega * a = (\iota\otimes\omega)\Delta(a)\in C(G).$$

Then, since $\omega = \omega_{\xi,\zeta}$ for some $\xi,\zeta\in H_0$, by definition of W we have

$$(\iota\otimes\omega)(W)a\xi_h = (\omega * a)\xi_h.$$

Therefore

$$(\iota\otimes\omega\otimes\nu)(\iota\otimes\Delta)(W)a\xi_h = (\iota\otimes\omega*\nu)(W)a\xi_h = ((\omega*\nu)*a)\xi_h,$$

and

$$(\iota\otimes\omega\otimes\nu)(W_{12}W_{13})a\xi_h = (\iota\otimes\omega)(W)(\iota\otimes\nu)(W)a\xi_h = (\omega*(\nu*a))\xi_h.$$

Hence $(\iota\otimes\Delta)(W) = W_{12}W_{13}$ by the coassociativity of Δ.

End of proof. If $C(G)$ is represented faithfully on another Hilbert space $\tilde H_0$, then the above arguments show that we can define a unitary $\tilde W\in M(K(L^2(G))\otimes C(G))$ such that $\tilde W(\Lambda(a)\otimes\xi) = \Delta(a)(\Lambda(1)\otimes\xi)$ for all $a\in C(G)$ and $\xi\in\tilde H_0$. Then $(\iota\otimes\omega)(\tilde W)a\xi_h = (\omega * a)\xi_h$ for any $\omega\in C(G)^*$ that extends to a normal linear functional on $B(\tilde H_0)$. Hence $(\iota\otimes\omega)(\tilde W) = (\iota\otimes\omega)(W)$ for all such ω, so $\tilde W = W$. Alternatively, we can give a space-free definition of W: it is an operator on the Hilbert $C(G)$-module $L^2(G)\otimes C(G)$ such that $W(a\xi_h\otimes 1) = \Delta(a)(\xi_h\otimes 1)$. $\square$

The representation W is called the right regular representation of G.

Theorem 1.5.3. — *For the right regular representation W of a compact quantum group G we have:*

(i) *the space* $\{(\omega\otimes\iota)(W) \mid \omega\in K(L^2(G))^*\}$ *is dense in* $C(G)$;

(ii) $W(\pi_h(a)\otimes 1)W^* = (\pi_h\otimes\iota)\Delta(a)$ *for all* $a\in C(G)$.

Proof. — Assume $C(G)\subset B(H_0)$, so $W(a\xi_h\otimes\xi)=\Delta(a)(\xi_h\otimes\xi)$ for $a\in C(G)$.

(i) Recall that we denote by $\omega_{\xi,\zeta}$ the linear functional defined by $\omega_{\xi,\zeta}(T)=(T\xi,\zeta)$. For any $a,b\in C(G)$ and $\zeta,\eta\in H$ we have

$$\begin{aligned}((\omega_{a\xi_h,b\xi_h}\otimes\iota)(W)\zeta,\eta) &= (W(a\xi_h\otimes\zeta),b\xi_h\otimes\eta)\\ &= (\Delta(a)(\xi_h\otimes\zeta),b\xi_h\otimes\eta)\\ &= ((h\otimes\iota)((b^*\otimes 1)\Delta(a))\zeta,\eta),\end{aligned}$$

so

$$(\omega_{a\xi_h,b\xi_h}\otimes\iota)(W) = (h\otimes\iota)((b^*\otimes 1)\Delta(a)).$$

Since $(A\otimes 1)\Delta(A)$ is dense in $C(G)\otimes C(G)$, we see that (i) holds.

(ii) For any $a,b\in C(G)$ we have

$$W(a\otimes 1)(b\xi_h\otimes\xi) = W(ab\xi_h\otimes\xi) = \Delta(ab)(\xi_h\otimes\xi) = \Delta(a)W(b\xi_h\otimes\xi),$$

so $W(\pi_h(a)\otimes 1) = (\pi_h\otimes\iota)(\Delta(a))W$. □

The following theorem extends Theorem 1.3.7 to infinite dimensional representations.

Theorem 1.5.4. — *Every unitary representation decomposes into a direct sum of finite dimensional irreducible unitary representations.*

Proof. — It is enough to show that any unitary representation

$$U\in M(K(H)\otimes C(G))$$

decomposes into a direct sum of finite dimensional representations. For $S\in K(H)$, consider the operator

$$T=(\iota\otimes h)(U^*(S\otimes 1)U)\in K(H).$$

Remark 1.3.6 applies also to infinite dimensional representations, so

$$U^*(T\otimes 1)U = T\otimes 1,$$

that is, $T\in\mathrm{End}(U)$. If we take a net of finite rank projections $p_i\in B(H)$ such that $p_i\nearrow 1$ strongly, then $(\iota\otimes h)(U^*(p_i\otimes 1)U)\nearrow 1$ strongly as well, since for any finite rank projection p we have $U^*(p_i\otimes 1)U(p\otimes 1)\to U^*U(p\otimes 1)=p\otimes 1$ in norm. Therefore $K(H)\cap\mathrm{End}(U)$ is a nondegenerate C*-algebra of compact operators on H. Hence H decomposes into a direct sum $H=\oplus_j e_jH$ for some finite rank projections $e_j\in\mathrm{End}(U)$. □

For a unitary representation $U \in M(K(H) \otimes C(G))$, the elements $(\omega_{\xi,\zeta} \otimes \iota)(U) \in C(G)$ for $\xi, \zeta \in H$ are called the matrix coefficients of U.

Corollary 1.5.5. — *The linear span of matrix coefficients of finite dimensional representations of G is dense in $C(G)$.*

Proof. — Consider the right regular representation $W \in M(K(L^2(G)) \otimes C(G))$. Decompose it into a direct sum of irreducible unitary representations:

$$L^2(G) = \oplus_i H_i, \quad W = \oplus_i U_i.$$

If $\xi \in H_i$ and $\eta \in H_j$, then for $\omega_{\xi,\eta} = (\cdot\,\xi, \eta)$ we have

$$(\omega_{\xi,\eta} \otimes \iota)(W) = \begin{cases} 0, & \text{if } i \neq j, \\ (\omega_{\xi,\eta} \otimes \iota)(U_i), & \text{if } i = j. \end{cases}$$

Since such functionals $\omega_{\xi,\eta}$ span a dense subspace in $K(L^2(G))^*$ and

$$\{(\omega \otimes \iota)(W) \mid \omega \in K(L^2(G))^*\}$$

is dense in $C(G)$ by Theorem 1.5.3, we see that the linear span of matrix coefficients of finite dimensional representations of G is dense in $C(G)$. □

A decomposition of the regular representation into irreducible representations can be described explicitly as follows. If U is an irreducible finite dimensional unitary representation, then for every vector $\zeta \in H_U$ we can define a map

$$H_U \to L^2(G), \quad \xi \mapsto (\dim_q U)^{1/2} (\omega_{\xi, \rho_U^{1/2}\zeta} \otimes \Lambda)(U).$$

It is easy to check that it intertwines U and W. By the orthogonality relations it is isometric if ζ is a unit vector. Furthermore, by the same relations, if we take mutually orthogonal vectors ζ and ζ' in H_U, then the corresponding images of H_U in $L^2(G)$ will be orthogonal. The space spanned by such images for all possible U (up to equivalence) is precisely the dense subspace of $L^2(G)$ spanned by matrix coefficients of finite dimensional representations. So we obtain a decomposition of W by fixing representatives U of equivalence classes of irreducible finite dimensional unitary representations and choosing orthonormal bases in H_U. In particular, every irreducible unitary representation U appears in W with multiplicity $\dim U$.

References. — [**97**].

1.6. HOPF $*$-ALGEBRA OF MATRIX COEFFICIENTS

For a compact quantum group G denote by $\mathbb{C}[G] \subset C(G)$ the linear span of matrix coefficients of all finite dimensional representations of G. This is a dense unital $*$-subalgebra of $C(G)$: the product of matrix coefficients is a matrix coefficient of the tensor product representation, the adjoint of a matrix coefficient is a matrix coefficient of the contragredient representation, and the density was proved in Corollary 1.5.5. Furthermore, $\Delta(\mathbb{C}[G])$ is contained in the algebraic tensor product $\mathbb{C}[G] \otimes \mathbb{C}[G] \subset C(G) \otimes C(G)$.

Definition 1.6.1. — A pair $(\mathscr{A}, \Delta)$ consisting of a unital $*$-algebra and a unital $*$-homomorphism $\Delta\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is called a Hopf $*$-algebra, if $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and there exist linear maps $\varepsilon\colon \mathscr{A} \to \mathbb{C}$ and $S\colon \mathscr{A} \to \mathscr{A}$ such that

$$(\varepsilon \otimes \iota)\Delta(a) = (\iota \otimes \varepsilon)\Delta(a) = a \;\text{ and }\; m(S \otimes \iota)\Delta(a) = m(\iota \otimes S)\Delta(a) = \varepsilon(a)1$$

for all $a \in \mathscr{A}$, where $m\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is the multiplication map.

The map ε is called counit and S is called antipode, or coinverse.

Example 1.6.2. — Assume G is a compact group. Then $\mathbb{C}[G]$ with the usual comultiplication, $\Delta(f)(g,h) = f(gh)$, is a Hopf $*$-algebra, with $\varepsilon(f) = f(e)$ and $S(f)(g) = f(g^{-1})$. $\square$

Example 1.6.3. — Assume Γ is a discrete group and let $G = \hat{\Gamma}$, so $C(G) = C^*_r(\Gamma)$ and $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma$. The elements $\lambda_\gamma \in C^*_r(\Gamma)$ are one-dimensional representations of G, and since they already span a dense subspace of $C(G)$, from the orthogonality relations we conclude that there are no other irreducible representations. Therefore $\mathbb{C}[G] \subset C^*_r(\Gamma)$ is the group algebra of Γ, spanned by the operators λ_γ. This is a Hopf $*$-algebra, with $\varepsilon(\lambda_\gamma) = 1$ and $S(\lambda_\gamma) = \lambda_{\gamma^{-1}}$. $\square$

Let us list a number of properties of the maps ε and S that follow from the axioms:

(a) ε and S are uniquely determined;

(b) $\varepsilon S = \varepsilon$;

(c) ε is a $*$-homomorphism and S is an anti-homomorphism;

(d) $\Delta S = (S \otimes S)\Delta^{\mathrm{op}}$;

(e) $S(S(a^*)^*) = a$ for all $a \in \mathscr{A}$.

We are not going to prove this, see for example [**84**]. Properties (b)-(e) will be obvious in our examples.

Theorem 1.6.4. — *For any compact quantum group G, the pair $(\mathbb{C}[G], \Delta)$ is a Hopf $*$-algebra.*

Proof. — We define the linear maps ε and S by letting

$$(\iota\otimes\varepsilon)(U)=1 \ \text{ and } \ (\iota\otimes S)(U)=U^{-1}$$

for any finite dimensional representation U of G. To see that such maps indeed exist, choose representatives U_α of equivalence classes of irreducible finite dimensional unitary representations. By the orthogonality relations the matrix coefficients u^α_{ij} of these representations with respect to fixed orthonormal bases in H_{U_α} form a linear basis in $\mathbb{C}[G]$. Therefore we can define $\varepsilon(u^\alpha_{ij})=\delta_{ij}$ and $S(u^\alpha_{ij})=u^{\alpha*}_{ji}$. Since any finite dimensional representation U is equivalent to a direct sum of copies of U_α, we then get $(\iota\otimes\varepsilon)(U)=1$ and $(\iota\otimes S)(U)=U^{-1}$.

By applying $\iota\otimes\varepsilon\otimes\iota$ to $(\iota\otimes\Delta)(U)=U_{12}U_{13}$ we get $(\varepsilon\otimes\iota)\Delta=\iota$, and by applying $\iota\otimes\iota\otimes\varepsilon$ we get $(\iota\otimes\varepsilon)\Delta=\iota$. On the other hand, by applying $(\iota\otimes m)(\iota\otimes S\otimes\iota)$ we get

$$(\iota\otimes m)(\iota\otimes S\otimes\iota)(\iota\otimes\Delta)(U)=(\iota\otimes m)(U_{12}^{-1}U_{13})=U^{-1}U=1=(\iota\otimes\varepsilon(\cdot)1)(U),$$

so $m(S\otimes\iota)\Delta=\varepsilon(\cdot)1$. Similarly, $m(\iota\otimes S)\Delta=\varepsilon(\cdot)1$. ☐

Although it took us a lot of effort to construct the Hopf $*$-algebra $(\mathbb{C}[G],\Delta)$ for a compact quantum group G, in practice $(\mathbb{C}[G],\Delta)$ is often known from the beginning and $C(G)$ is defined as a C*-completion of $\mathbb{C}[G]$. It is therefore important to have a characterization of Hopf $*$-algebras that arise from compact quantum groups. We will need a few definitions to formulate the result.

Definition 1.6.5. — A corepresentation of a Hopf $*$-algebra $(\mathscr{A},\Delta)$ on a vector space H is a linear map $\delta\colon H\to H\otimes\mathscr{A}$ such that

$$(\delta\otimes\iota)\delta=(\iota\otimes\Delta)\delta \ \text{ and } \ (\iota\otimes\varepsilon)\delta=\iota.$$

The corepresentation is called unitary if H is a Hilbert space and

$$\langle\delta(\xi),\delta(\zeta)\rangle=(\xi,\zeta)1 \ \text{ for all } \ \xi,\zeta\in H,$$

where $\langle\cdot,\cdot\rangle$ is defined by $\langle\xi\otimes a,\zeta\otimes b\rangle=(\xi,\zeta)b^*a\in\mathscr{A}$.

A subspace $K\subset H$ is called invariant if $\delta(K)\subset K\otimes\mathscr{A}$. The corepresentation is called irreducible if there are no proper invariant subspaces.

If $\delta\colon H\to H\otimes\mathscr{A}$ is a corepresentation on a finite dimensional space, then $\delta(\xi)=U(\xi\otimes1)$ for a uniquely defined element $U\in B(H)\otimes\mathscr{A}$. Namely, if $\{\xi_i\}_{i=1}^n$ is a basis in H and $\delta(\xi_j)=\sum_i\xi_i\otimes u_{ij}$, then $U=\sum_{i,j}m_{ij}\otimes u_{ij}$. Furthermore, we have

$$(\iota\otimes\Delta)(U)=U_{12}U_{13} \ \text{ and } \ (\iota\otimes\varepsilon)(U)=1.$$

Conversely, any element $U\in B(H)\otimes\mathscr{A}$ with the above properties defines a corepresentation, so from now on we will use both pictures interchangeably. In particular, by matrix coefficients of δ we mean the matrix coefficients of U.

All our principal results on finite dimensional representations of compact quantum groups can be extended to finite dimensional corepresentations of Hopf $*$-algebras. Moreover, as we will see now, the proofs become easier due to the existence of the antipode.

Applying $\iota \otimes m(\iota \otimes S)$ and $\iota \otimes m(S \otimes \iota)$ to $(\iota \otimes \Delta)(U) = U_{12}U_{13}$ we see that U is invertible and $(\iota \otimes S)(U) = U^{-1}$.

Unitarity of δ is equivalent to $U^*U = 1$, but since U is invertible, this is the same as unitarity of U. In particular, we see that a finite dimensional unitary corepresentation of $(\mathbb{C}[G], \Delta)$ is the same thing as a finite dimensional unitary representation of G.

If δ is irreducible, then there are no nonscalar operators $T \in B(H)$ such that $T \otimes 1$ commutes with U, since any eigenspace of such an operator would be an invariant subspace of H. Therefore for $(\mathbb{C}[G], \Delta)$ the notion of irreducibility of corepresentations is formally stronger than the notion of irreducibility of representations of G. The two notions nevertheless coincide, since a subspace $K \subset H$ is invariant if and only if it is invariant for the subalgebra of $B(H)$ spanned by the operators $(\iota \otimes \omega)(U)$, $\omega \in \mathbb{C}[G]^*$, and as we remarked after Proposition 1.3.9 for irreducible representations this algebra coincides with $B(H)$.

Lemma 1.6.6. — *For any corepresentation $\delta\colon H \to H \otimes \mathscr{A}$, finite or infinite dimensional, we have $H \otimes \mathscr{A} = \delta(H)(1 \otimes \mathscr{A})$.*

Proof. — Consider the linear maps $s, r\colon H \otimes \mathscr{A} \to H \otimes \mathscr{A}$ defined by

$$s(\xi \otimes a) = (\iota \otimes S)\delta(\xi)(1 \otimes a), \quad r(\xi \otimes a) = \delta(\xi)(1 \otimes a).$$

We claim that $rs = \iota$. Since both maps are right $\mathscr{A}$-module maps, it suffices to compute rs on $\xi \otimes 1$. We have

$$\begin{aligned} rs(\xi \otimes 1) &= r((\iota \otimes S)\delta(\xi)) = (\iota \otimes m(\iota \otimes S))(\delta \otimes \iota)\delta(\xi) \\ &= (\iota \otimes m(\iota \otimes S))(\iota \otimes \Delta)\delta(\xi) \\ &= (\iota \otimes \varepsilon(\cdot)1)\delta(\xi) = \xi \otimes 1. \end{aligned}$$

Therefore $rs = \iota$, so r is surjective. It is also easy to check that r is injective, but we do not need this. □

This lemma implies in particular that if δ is unitary and $K \subset H$ is a closed invariant subspace, then $K^\perp$ is also invariant. Indeed, for $\xi \in K^\perp$, $\zeta \in K$ and $a \in \mathscr{A}$ we have

$$\langle \delta(\xi), \delta(\zeta)(1 \otimes a) \rangle = (\xi, \zeta)a^* = 0.$$

Since $\delta(K)(1 \otimes \mathscr{A}) = K \otimes \mathscr{A}$, we therefore get $\langle \delta(\xi), \zeta \otimes 1 \rangle = 0$ for all $\zeta \in K$, but this exactly means that $\delta(\xi) \in K^\perp \otimes \mathscr{A}$. It follows that any finite dimensional unitary corepresentation decomposes into a direct sum of finite dimensional irreducible unitary corepresentations.

The same conclusion can also be obtained by the following argument, which we will be important later. Consider the dual space $\mathscr{U} = \mathscr{A}^*$. This is a unital $*$-algebra with product and involution given by

$$\omega\nu = (\omega \otimes \nu)\Delta, \quad \omega^* = \bar{\omega}S,$$

and with unit ε; here $\bar{\omega}$ is defined by $\bar{\omega}(a) = \overline{\omega(a^*)}$. Any finite dimensional corepresentation $U \in B(H) \otimes \mathscr{A}$ defines a unital representation $\pi_U : \mathscr{U} \to B(H)$ by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. A subspace $K \subset H$ is invariant for δ if and only if it is $\pi_U(\mathscr{U})$-invariant. Now, if U is unitary, then π_U is a $*$-representation, since

$$\pi_U(\omega)^* = (\iota \otimes \omega)(U)^* = (\iota \otimes \bar{\omega})(U^*)$$

and $U^* = (\iota \otimes S)(U)$. Hence, if $K \subset H$ is $\pi_U(\mathscr{U})$-invariant, then $K^\perp$ is also $\pi_U(\mathscr{U})$-invariant.

We are now ready to formulate and prove a characterization of the Hopf $*$-algebras $(\mathbb{C}[G], \Delta)$.

Theorem 1.6.7. — *Assume $(\mathscr{A}, \Delta)$ is a Hopf $*$-algebra such that $\mathscr{A}$ is generated as an algebra by matrix coefficients of finite dimensional unitary corepresentations. Then $(\mathscr{A}, \Delta) = (\mathbb{C}[G], \Delta)$ for a compact quantum group G.*

Proof. — Consider the C*-enveloping algebra A of $\mathscr{A}$. It is well-defined, since $\mathscr{A}$ is generated by matrix coefficients of unitary matrices over $\mathscr{A}$, and these have universal bounds on the norms for all possible $*$-representations on Hilbert spaces. The crux of the matter is to show that the canonical homomorphism $\mathscr{A} \to A$ is injective. For this we will show that there exists a faithful state h on $\mathscr{A}$ that plays the role of the Haar state. We will construct h in several steps.

Step 1. There exists a unique linear functional h such that $h(1) = 1$, $(\iota \otimes h)\Delta(a) = h(a)1$ and $(h \otimes \iota)\Delta(a) = h(a)1$ for all $a \in \mathscr{A}$.

Observe first that if $U_1 \in B(H_1) \otimes \mathscr{A}, \dots, U_n \in B(H_n) \otimes \mathscr{A}$ are pairwise nonequivalent finite dimensional irreducible corepresentations, then their matrix coefficients with respect to fixed bases in $H_1, \dots, H_n$ are linearly independent. Indeed, the representations $\pi_{U_1}, \dots, \pi_{U_n}$ of $\mathscr{U}$ are irreducible and pairwise nonequivalent. Hence, by Jacobson's density theorem, see e.g., [**58**, Theorem XVII.3.2], the homomorphism $\oplus_i \pi_{U_i} : \mathscr{U} \to \oplus_i B(H_i)$ is surjective. For dimension reasons this is possible only if the matrix coefficients of $U_1, \dots, U_n$ are linearly independent.

By assumption $\mathscr{A}$ is generated by matrix coefficients of finite dimensional unitary corepresentations. The product of matrix coefficients is a matrix coefficient of the tensor product of corepresentations, defined as in the quantum group case by $U \times V = U_{13}V_{23}$. Since any finite dimensional unitary corepresentation decomposes into a direct sum of irreducible ones, it follows that $\mathscr{A}$ is spanned by matrix coefficients of finite dimensional irreducible unitary corepresentations. Choose representatives U_α of the equivalence classes of finite dimensional irreducible unitary corepresentations of $\mathscr{A}$. By the above observation the matrix coefficients of U_α (with respect to any fixed bases in H_{U_α}) form a basis in $\mathscr{A}$. We can therefore define a linear functional h on $\mathscr{A}$ such that $h(1) = 1$ and $(\iota \otimes h)(U_\alpha) = 0$ if $U_\alpha \neq 1$. Since $(\iota \otimes \Delta)(U_\alpha) = (U_\alpha)_{12}(U_\alpha)_{13}$, the functional h has the required properties. The uniqueness is obvious.

Step 2. The following orthogonality relations hold: for every α there exists a positive invertible operator $Q_\alpha \in B(\bar{H}_{U_\alpha})$ such that

$$(\iota \otimes h)(U_\alpha^*(T \otimes 1)U_\beta) = \delta_{\alpha,\beta}\frac{\mathrm{Tr}(Tj(Q_\alpha))}{\dim U_\alpha}1 \text{ for all } T \in B(H_{U_\beta}, H_{U_\alpha}).$$

The key point to observe is that our proof of the orthogonality relations, Theorem 1.4.3, did not use the positivity of the Haar state in any way. Namely, that proof shows that the above relations hold with $Q_\alpha = (\iota \otimes h)(U_\alpha^{c*}U_\alpha^c)$, where $U_\alpha^c = (j \otimes \iota)(U_\alpha^{-1}) = (j \otimes S)(U_\alpha)$ is the contragredient corepresentation to U_α. We thus only have to check that Q_α is positive and invertible for every α.

In order to show this, note that by the proof of Lemma 1.3.12(i) we have $j(Q_\alpha) \in \mathrm{Mor}(U_\alpha, U_\alpha^{cc})$. But by irreducibility of U_α the space $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ is at most one-dimensional, and every nonzero operator there is invertible. Since $\mathrm{Tr}(j(Q_\alpha)) = \dim U_\alpha > 0$, it is therefore enough to show that $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ contains a nonzero positive operator.

The corepresentation U_α^c is irreducible, because $\pi_{U_\alpha^c}(\omega) = j(\pi_{U_\alpha}(\omega S))$ and S is bijective. As we observed in Step 1, the matrix coefficients of pairwise nonequivalent finite dimensional irreducible corepresentations are linearly independent. It follows that there exists a finite dimensional irreducible unitary corepresentation V among U_β that is equivalent to U_α^c. Choose an invertible operator $T \in \mathrm{Mor}(U_\alpha^c, V)$. Then $j(T) \in \mathrm{Mor}(V^c, U_\alpha^{cc})$. On the other hand, starting with the identity

$$(T \otimes 1)U_\alpha^c = V(T \otimes 1)$$

and taking the adjoints and then applying $j \otimes \iota$ we get

$$(j(T)^* \otimes 1)U_\alpha = (j \otimes \iota)(V^*)(j(T)^* \otimes 1) = V^c(j(T)^* \otimes 1),$$

so $j(T)^* \in \mathrm{Mor}(U_\alpha, V^c)$. Therefore $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ contains the positive invertible operator $j(T)j(T)^*$.

Step 3. We have $h(a^*a) > 0$ for all $a \in \mathscr{A}$, $a \neq 0$.

Let u_{ij}^α be the matrix coefficients of U_α with respect to orthonormal bases in which the positive invertible operators $j(Q_\alpha)$ are diagonal. By Step 2 these matrix coefficients form an orthogonal basis in $\mathscr{A}$ with respect to the sesquilinear form $(a, b) = h(b^*a)$, and $(u_{ij}^\alpha, u_{ij}^\alpha) > 0$. Hence this form is positive definite.

End of proof. The left action of $\mathscr{A}$ on itself gives us a faithful $*$-representation of $\mathscr{A}$ on the pre-Hilbert space $\mathscr{A}$ equipped with the scalar product $(a,b) = h(b^*a)$. This is a representation by bounded operators, since $\mathscr{A}$ is spanned by matrix coefficients of unitary matrices over $\mathscr{A}$ and every entry of a unitary matrix must act as an operator of norm not larger than 1. Hence this representation extends to a faithful representation on the Hilbert space completion of $\mathscr{A}$. Therefore $\mathscr{A}$ can be considered as a subalgebra of its C*-enveloping algebra A. The homomorphism Δ extends to a unital $*$-homomorphism $A \to A \otimes A$, which we continue to denote by Δ. By Lemma 1.6.6 applied to $\delta = \Delta$ we have $(1 \otimes \mathscr{A})\Delta(\mathscr{A}) = \mathscr{A} \otimes \mathscr{A}$, and similarly one proves that $(\mathscr{A} \otimes 1)\Delta(\mathscr{A}) = \mathscr{A} \otimes \mathscr{A}$. It follows that (A, Δ) has the cancellation property, so it is a compact quantum group G. Clearly, $\mathscr{A} \subset \mathbb{C}[G]$. Since $\mathscr{A}$ is dense in $\mathbb{C}[G]$ and is spanned by matrix coefficients of irreducible unitary representations of G, by the orthogonality relations for G we conclude that $\mathscr{A} = \mathbb{C}[G]$. □

If $\mathscr{A}$ is a finite dimensional Hopf $*$-algebra, then the $*$-algebra $\mathscr{U}$ we introduced before Theorem 1.6.7 is in fact a Hopf $*$-algebra with comultiplication

$$\hat{\Delta}(\omega)(a \otimes b) = \omega(ab) \text{ for } a, b \in \mathscr{A}.$$

The antipode is given by $\hat{S}(\omega) = \omega S$ and the counit by $\hat{\varepsilon}(\omega) = \omega(1)$. The axioms are verified via a straightforward computation. The Hopf $*$-algebra $(\mathscr{U}, \hat{\Delta})$ is called the dual of $(\mathscr{A}, \Delta)$. By taking the dual of $(\mathscr{U}, \hat{\Delta})$ we get back $(\mathscr{A}, \Delta)$.

Even if $\mathscr{A}$ is not finite dimensional, we still have a structure on $\mathscr{U}$ reminiscent of a Hopf $*$-algebra. We will define it in the case $\mathscr{A} = \mathbb{C}[G]$, although the construction makes sense in general. Let us first introduce some notation.

Definition 1.6.8. — The algebra of functions on the dual discrete quantum group $\hat{G}$ of a compact quantum group G is the $*$-algebra $\mathscr{U}(G) = \mathbb{C}[G]^*$ with multiplication and involution given by

$$\omega\nu = \omega * \nu = (\omega \otimes \nu)\Delta \text{ and } \omega^* = \bar{\omega}S.$$

Example 1.6.9. — Assume Γ is a discrete group and let $G = \hat{\Gamma}$. As we showed in Example 1.6.3, in this case $\mathbb{C}[G]$ is simply the group algebra of Γ. Then $\mathscr{U}(G)$ is the usual algebra of functions on Γ, with pointwise multiplication and involution. □

For every finite dimensional representation U of G we defined a representation $\pi_U \colon \mathscr{U}(G) \to B(H_U)$ by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. It is $*$-preserving if U is unitary. Fix representatives U_α of the equivalence classes of irreducible unitary representations of G. Then $\mathbb{C}[G]$ is the direct sum of the spaces spanned by the matrix coefficients of U_α. It follows that the homomorphisms π_{U_α} define a $*$-isomorphism

$$\mathscr{U}(G) \cong \prod_\alpha B(H_{U_\alpha}).$$

We will use the notation $\mathscr{U}(G^n)$ for the dual space of $\mathbb{C}[G]^{\otimes n}$. Similarly to $\mathscr{U}(G)$, this is a $*$-algebra, which is canonically isomorphic to

$$\prod_{\alpha_1,\dots\alpha_n} B(H_{U_{\alpha_1}}\otimes\cdots\otimes H_{U_{\alpha_n}}).$$

Define a map $\hat{\Delta}\colon \mathscr{U}(G)\to\mathscr{U}(G\times G)$ by

$$\hat{\Delta}(\omega)(a\otimes b)=\omega(ab)\ \text{ for }\ a,b\in\mathbb{C}[G].$$

It is a unital $*$-homomorphism. Equivalently, $\hat{\Delta}(\omega)\in\mathscr{U}(G\times G)$ is the unique element such that

$$(\pi_{U_\alpha}\otimes\pi_{U_\beta})\hat{\Delta}(\omega)T=T\pi_{U_\gamma}(\omega)\ \text{ for all }\ T\in\mathrm{Mor}(U_\gamma,U_\alpha\times U_\beta).$$

Note that in general the image of $\hat{\Delta}$ is not contained in the algebraic tensor product $\mathscr{U}(G)\otimes\mathscr{U}(G)\subset\mathscr{U}(G\times G)$; indeed, this is not the case already for $G=\hat{\Gamma}$ when the discrete group Γ is infinite.

Define also maps $\hat{\varepsilon}\colon\mathscr{U}(G)\to\mathbb{C}$ and $\hat{S}\colon\mathscr{U}(G)\to\mathscr{U}(G)$ by

$$\hat{\varepsilon}(\omega)=\omega(1)\ \text{ and }\ \hat{S}(\omega)=\omega S.$$

The maps $\hat{\Delta}$, $\hat{\varepsilon}$, $\hat{S}$, as well as the multiplication map $m\colon\mathscr{U}(G)\otimes\mathscr{U}(G)\to\mathscr{U}(G)$, can be applied to factors of $\mathscr{U}(G)^{\otimes n}$ and then extended to $\mathscr{U}(G^n)$. For example, the map $\iota\otimes\hat{\Delta}\colon\mathscr{U}(G^2)\to\mathscr{U}(G^3)$ is defined by

$$(\iota\otimes\hat{\Delta})(\omega)(a\otimes b\otimes c)=\omega(a\otimes bc).$$

With this understanding, one can now easily check that the pair $(\mathscr{U}(G),\hat{\Delta})$ satisfies all the axioms of a Hopf $*$-algebra.

The pair $(\mathscr{U}(G),\hat{\Delta})$ will play an important role in the book, in many respects more important than the pair $(C(G),\Delta)$. The properties of $(\mathscr{U}(G),\hat{\Delta})$ can be formalized, leading to a theory of discrete quantum groups that provides a dual picture for compact quantum groups [**85, 86**].

References. — [**24**], [**84**], [**85**], [**86**], [**97**].

1.7. MODULAR PROPERTIES OF THE HAAR STATE

Let G be a compact quantum group. Recall that in Section 1.4 we introduced the positive invertible operators $\rho_U\in\mathrm{Mor}(U,U^{cc})$.

Definition 1.7.1. — The Woronowicz characters is the family $\{f_z\}_{z\in\mathbb{C}}$ of linear functionals on $\mathbb{C}[G]$ defined by

$$(\iota\otimes f_z)(U)=\rho_U^z$$

for all finite dimensional unitary representations U of G.

That the functionals f_z are well-defined can be checked using the same arguments as the ones used in the proof of Theorem 1.6.4 to show that ε is well-defined. Note that $f_0=\varepsilon$. The functional f_1 will be denoted by ρ. Since the dual space $\mathscr{U}(G)=\mathbb{C}[G]^*$ is a $*$-algebra isomorphic to a product of full matrix algebras, the functional calculus makes sense in $\mathscr{U}(G)$. Then

$$f_z=\rho^z\in\mathscr{U}(G).$$

By Theorem 1.4.9 we have $\rho_{U\times V}=\rho_U\otimes\rho_V$, which means that

$$\hat\Delta(\rho)=\rho\otimes\rho,\ \text{ hence also }\ \hat S(\rho)=\rho^{-1},$$

since $m(\iota\otimes\hat S)\hat\Delta(\rho)=\hat\varepsilon(\rho)1=1$. Note in passing that the identity $\hat S(\rho)=\rho^{-1}$ can also be deduced from Proposition 1.4.7.

Proposition 1.7.2. — *The linear functionals f_z have the following properties:*

(i) f_z *is a homomorphism* $\mathbb{C}[G]\to\mathbb{C}$*;*

(ii) $\bar f_z=f_{-\bar z}$*;*

(iii) $f_{z_1}*f_{z_2}=f_{z_1+z_2}$.

Proof. — Part (i) follows from $\hat\Delta(\rho^z)=\rho^z\otimes\rho^z$, while (iii) from $\rho^{z_1}\rho^{z_2}=\rho^{z_1+z_2}$. To prove (ii) note that for any $\omega\in\mathscr{U}(G)$ we have $\bar\omega=\hat S(\omega)^*$, since

$$\hat S(\omega)^*(a)=\overline{\hat S(\omega)(S(a)^*)}=\overline{\omega(S(S(a)^*))}=\overline{\omega(a^*)}$$

for any $a\in\mathbb{C}[G]$. As $\hat\Delta(\rho^z)=\rho^z\otimes\rho^z$, we also have $\hat S(\rho^z)=\rho^{-z}$. Hence

$$\bar f_z=\hat S(\rho^z)^*=(\rho^{-z})^*=\rho^{-\bar z}=f_{-\bar z}.\qquad\square$$

Since f_z are multiplicative, the maps

$$a\mapsto a*f_z=(f_z\otimes\iota)\Delta(a)\ \text{ and }\ a\mapsto f_z*a=(\iota\otimes f_z)\Delta(a)$$

are homomorphisms $\mathbb{C}[G]\to\mathbb{C}[G]$. Put

$$\sigma_z(a)=f_{iz}*a*f_{iz}.$$

In other words, for any finite dimensional unitary representation U of G we have

$$(\iota\otimes\sigma_z)(U)=(\rho_U^{iz}\otimes1)U(\rho_U^{iz}\otimes1).$$

The map $\mathbb{C}\ni z\mapsto\sigma_z\in\mathrm{Aut}(\mathbb{C}[G])$ is a homomorphism, since $f_{z_1}*f_{z_2}=f_{z_1+z_2}$. We also have $\sigma_z(a^*)=\sigma_{\bar z}(a)^*$, because $\bar f_{iz}=f_{i\bar z}$. In particular, $(\sigma_t)_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of $\mathbb{C}[G]$, called the modular group. The name is justified by the following result, proved by a straightforward computation using the orthogonality relations.

Theorem 1.7.3. — *The Haar state h on $\mathbb{C}[G]$ is σ_z-invariant for all $z\in\mathbb{C}$, and $h(ab)=h(b\sigma_{-i}(a))$ for all $a,b\in\mathbb{C}[G]$.*

Example 1.7.4. — Consider the quantum group $SU_q(2)$ and its fundamental representation $U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. By Example 1.4.2 we have $\rho_U = \begin{pmatrix} |q|^{-1} & 0 \\ 0 & |q| \end{pmatrix}$. It follows that the modular group is defined by

$$\sigma_t(U) = \begin{pmatrix} |q|^{-it} & 0 \\ 0 & |q|^{it} \end{pmatrix} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} |q|^{-it} & 0 \\ 0 & |q|^{it} \end{pmatrix},$$

so $\sigma_t(\alpha) = |q|^{-2it}\alpha$, $\sigma_t(\gamma) = \gamma$. □

Denote by $C_r(G)$ the image of $C(G)$ under the GNS-representation $\pi_h\colon C(G) \to B(L^2(G))$. The automorphisms σ_t define a strongly continuous one-parameter group of $*$-automorphisms of $C_r(G)$, which we continue to denote by σ_t. The cyclic vector $\xi_h \in L^2(G)$ defines a state on $C_r(G)$, which we also denote by the same symbol h.

Corollary 1.7.5. — *The state h on $C_r(G)$ is faithful.*

Proof. — By Theorem 1.7.3 the state h on $C_r(G)$ is a σ-KMS$_{-1}$-state. A basic result on KMS-states asserts that the cyclic vector in the GNS-representation of a KMS-state is separating, see [**16**, Corollary 5.3.9]. □

As we already observed in the proof of Theorem 1.6.7, the orthogonality relations imply that $h(a^*a) > 0$ for all $a \in \mathbb{C}[G]$, $a \neq 0$. Hence the GNS-representation π_h is faithful on $\mathbb{C}[G]$. Therefore $C_r(G)$ is a completion of $\mathbb{C}[G]$. The homomorphism $\Delta\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ extends to a homomorphism $\Delta_r\colon C_r(G) \to C_r(G) \otimes C_r(G)$. Indeed, by Theorem 1.5.3 we have

$$\Delta_r(a) = W_r(a \otimes 1)W_r^*,$$

where $W_r = (\iota \otimes \pi_h)(W) \in B(L^2(G) \otimes L^2(G))$ and $W \in M(K(L^2(G)) \otimes C(G))$ is the right regular representation of G. Since $\mathbb{C}[G]$ is dense in $C_r(G)$, by the orthogonality relations the algebra of matrix coefficients of $(C_r(G), \Delta_r)$ must coincide with $\mathbb{C}[G]$.

Therefore starting from $(C(G), \Delta)$ we get a new compact quantum group $(C_r(G), \Delta_r)$ with the same representation theory and a faithful Haar state. It is called the reduced form of $(C(G), \Delta)$.

In addition to the reduced form we can define a universal form $(C_u(G), \Delta_u)$. The C*-algebra $C_u(G)$ is defined as the C*-enveloping algebra of $\mathbb{C}[G]$. This algebra is well-defined, since $\mathbb{C}[G]$ is spanned by matrix coefficients of unitary matrices over $\mathbb{C}[G]$, and these have universal bounds on the norms for all possible $*$-representations. The homomorphism $\Delta\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ extends to a homomorphism $\Delta_u\colon C_u(G) \to C_u(G) \otimes C_u(G)$ by universality. By universality we also have a surjective homomorphism $C_u(G) \to C(G)$.

Therefore we have a sequence of surjective homomorphisms $C_u(G) \to C(G) \to C_r(G)$, with $\mathbb{C}[G]$ sitting densely in all three algebras. We will discuss the question when $C_u(G) \to C_r(G)$ is an isomorphism in Section 2.7.

In addition to the modular group $(\sigma_t)_t$ of the Haar state, the element $\rho \in \mathscr{U}(G)$ can be used to define another important family of automorphisms that provides an analytic continuation of the even powers of the antipode. Before we introduce it, let us first compute S^2.

Proposition 1.7.6. — *For any finite dimensional unitary representation U of G we have*

$$(\iota \otimes S^2)(U) = (\rho_U \otimes 1)U(\rho_U^{-1} \otimes 1).$$

Proof. — Recall that S was defined by $(\iota \otimes S)(U) = U^{-1}$ for finite dimensional representations U. This can be written as $(j \otimes S)(U) = U^c$. Hence

$$(\iota \otimes S^2)(U) = U^{cc}.$$

By definition, if U is unitary, we have $\rho_U \in \mathrm{Mor}(U, U^{cc})$, so

$$(\iota \otimes S^2)(U)(\rho_U \otimes 1) = (\rho_U \otimes 1)U. \qquad \square$$

Remark 1.7.7. — An equivalent formulation of the above proposition is that for any $\omega \in \mathscr{U}(G)$ we have $\hat{S}^2(\omega) = \rho\omega\rho^{-1}$. Furthermore, as one can easily check, ρ is the only positive invertible element in $\mathscr{U}(G)$ with this property and such that $\mathrm{Tr}(\pi_U(\rho)) = \mathrm{Tr}(\pi_U(\rho)^{-1})$ for any irreducible representation U of G.

Now for $z \in \mathbb{C}$ define an automorphism τ_z of $\mathbb{C}[G]$ by

$$\tau_z(a) = f_{-iz} * a * f_{iz}.$$

In other words,

$$(\iota \otimes \tau_z)(U) = (\rho_U^{iz} \otimes 1)U(\rho_U^{-iz} \otimes 1).$$

We then see that $S^2 = \tau_{-i}$. The one-parameter group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathbb{C}[G]$ is called the scaling group.

Example 1.7.8. — A computation similar to the one in Example 1.7.4 shows that for $G = SU_q(2)$ we have $\tau_t(\alpha) = \alpha$, $\tau_t(\gamma) = |q|^{2it}\gamma$. $\square$

Proposition 1.7.9. — *The following conditions are equivalent:*

(i) $\rho = 1$;

(ii) *the Haar state h is a trace;*

(iii) $S^2 = \iota$;

(iv) S *is $*$-preserving.*

Proof. — (i) $\Rightarrow$ (ii), (iii) This is clear, since σ_z and τ_z are trivial when $\rho = 1$.

(ii) $\Rightarrow$ (i) If h is trace, then $h(ab) = h(ba) = h(a\sigma_{-i}(b))$ for all $a, b \in \mathbb{C}[G]$. Since h is faithful on $\mathbb{C}[G]$, we conclude that $\sigma_{-i} = \iota$. By definition of σ_{-i} this implies that $\rho\omega\rho = \omega$ for all $\omega \in \mathscr{U}(G)$. Since ρ is positive, this is possible only when $\rho = 1$.

(iii) $\Rightarrow$ (i) If $S^2 = \iota$, then from Proposition 1.7.6 we see that $\rho_U \in \mathrm{End}(U)$. Hence ρ_U is scalar for irreducible U, and then $\rho_U = 1$ by the normalization condition on ρ_U.

(iii) $\Leftrightarrow$ (iv) This follows from $S(a)^* = S^{-1}(a^*)$. $\square$

Under the equivalent conditions of the above proposition the compact quantum group G is said to be of Kac type.

Note that if G is a compact matrix pseudogroup, so that the C*-algebra $C(G)$ is generated by matrix coefficients of one finite dimensional unitary representation U, then G is of Kac type if and only if $\rho_U = 1$. Indeed, the $*$-algebra generated by the matrix coefficients of U is the linear span of matrix coefficients of tensor products of copies of U and U^c. By the orthogonality relations it follows that any irreducible unitary representation of G is equivalent to a subrepresentation of a tensor product of copies of U and U^c. Hence, if $\rho_U = 1$, then $\rho_V = 1$ for any irreducible unitary representation V.

Example 1.7.10. — As follows from Example 1.4.2, the quantum group $SU_q(2)$ is not of Kac type for $q \neq \pm 1$. Similarly, the quantum groups $A_u(F)$ and $A_o(F)$ are typically not of Kac type. On the other hand, $A_u(n)$, $A_o(n)$, $A_s(n)$, genuine compact groups, as well as the duals $G = \hat{\Gamma}$ of discrete groups, are of Kac type. $\square$

One more conclusion we can draw from the above considerations is that the map S is unbounded on $\mathbb{C}[G] \subset C(G)$ unless G is of Kac type. Indeed, take an irreducible unitary representation U such that $\rho_U \neq 1$. Since $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$, it follows that the spectrum of ρ_U contains a number > 1 and a number < 1. Hence, for every $n \in \mathbb{N}$, we can find $m \in \mathbb{N}$ such that for an irreducible unitary subrepresentation V of $U^{\times m}$ the operator ρ_V has an eigenvalue $\lambda_1 > n$. It also has an eigenvalue $\lambda_2 < 1$. Then under the action of S^2 the matrix coefficient of V defined by the corresponding eigenvectors gets multiplied by $\lambda_1\lambda_2^{-1} > n$. Hence the map S^2 is unbounded.

When G is not of Kac type and so S is not $*$-preserving, it is sometimes convenient to consider a sort of polar decomposition of S. The automorphism $\tau_{-i/2}$ plays the role of the absolute value, and the map $R = S\tau_{i/2}$, called the unitary antipode, plays the role of the unitary part. For finite dimensional unitary representations we have

$$(j \otimes S)(U) = U^c \quad \text{and} \quad (j \otimes R)(U) = \bar{U}. \tag{1.7.1}$$

Proposition 1.7.11. — *The unitary antipode R is an involutive $*$-anti-automorphism of $\mathbb{C}[G]$ such that $\Delta R = (R \otimes R)\Delta^{\mathrm{op}}$.*

Proof. — Using that $\tau_z(a^*) = \tau_{\bar{z}}(a)^*$ and $(\iota \otimes S)(U) = U^*$ for unitary representations U, a straightforward computation yields $S\tau_z = \tau_z S$. In particular, $R = \tau_{i/2}S$, hence

$$R^2 = \tau_{i/2}S^2\tau_{i/2} = \iota.$$

This also follows from $(j \otimes R)(U) = \bar{U}$, since $\bar{\bar{U}} = U$.

Since $\tau_{i/2}$ is a homomorphism and S is an anti-homomorphism, R is an anti-homomorphism.

As $S = \tau_{-i/2}R = R\tau_{-i/2}$ and so $S(a)^* = \tau_{-i/2}(R(a))^* = \tau_{i/2}(R(a)^*)$, using $a^* = S(S(a)^*)$ we get

$$a^* = (R\tau_{-i/2})(\tau_{i/2}(R(a)^*)) = R(R(a)^*).$$

Hence $R(a^*) = R(a)^*$.

Finally, since $\Delta S = (S \otimes S)\Delta^{\mathrm{op}}$ and $\Delta\tau_z = (\tau_z \otimes \tau_z)\Delta$, which is immediate from $(\iota \otimes \tau_z)(U) = (\rho_U^{iz} \otimes 1)U(\rho_U^{-iz} \otimes 1)$, we get $\Delta R = (R \otimes R)\Delta^{\mathrm{op}}$. □

The unitary antipode on $\mathscr{U}(G)$ is defined by $\hat{R}(\omega) = \rho^{-1/2}\hat{S}(\omega)\rho^{1/2} = \omega R$. It is an involutive $*$-anti-automorphism of $\mathscr{U}(G)$ such that $\hat{\Delta}\hat{R} = (\hat{R} \otimes \hat{R})\hat{\Delta}^{\mathrm{op}}$. This is even easier to verify than to check the properties of R.

References. — [**55**], [**63**], [**94**], [**97**].

CHAPTER 2

C*-TENSOR CATEGORIES

In this chapter we study compact quantum groups from a categorical point of view. The central result is an extension of the Tannaka-Krein duality to the quantum setting. It shows how a compact quantum group can be reconstructed from its representation category concretely realized as a category of Hilbert spaces and, furthermore, characterizes categories that arise from representations of compact quantum groups. In addition to proving this general quite abstract result, we discuss what it means for categories of representations of Hopf $*$-algebras. This in particular allows us to look from a different angle at some of the examples from the previous chapter and generalize, using the Drinfeld-Jimbo quantized universal enveloping algebras, the deformation $SU_q(2)$ of $SU(2)$ to all simply connected semisimple compact Lie groups.

2.1. BASIC DEFINITIONS

The following definition is discouragingly long, but the reader should keep in mind that it simply tries to capture the essential properties of the category of Hilbert spaces with bounded linear operators as morphisms.

Definition 2.1.1. — A category $\mathscr{C}$ is called a C*-category if

(i) $\mathrm{Mor}(U,V)$ is a Banach space for all objects U and V, the map

$$\mathrm{Mor}\,(V,W) \times \mathrm{Mor}\,(U,V) \to \mathrm{Mor}\,(U,W), \quad (S,T) \mapsto ST,$$

is bilinear, and $\|ST\| \le \|S\|\,\|T\|$;

(ii) we are given an antilinear contravariant functor $*\colon \mathscr{C} \to \mathscr{C}$ that is the identity map on objects, so if $T \in \mathrm{Mor}(U,V)$, then $T^* \in \mathrm{Mor}(V,U)$, and that satisfies the following properties:

(a) $T^{**} = T$ for any morphism T;

(b) $\|T^*T\| = \|T\|^2$ for any $T \in \mathrm{Mor}(U,V)$; in particular, $\mathrm{End}(U) = \mathrm{Mor}(U,U)$ is a unital C*-algebra for every object U;

(c) for any morphism $T \in \mathrm{Mor}(U,V)$, the element T^*T of the C*-algebra $\mathrm{End}(U)$ is positive.

Condition (c) is sometimes omitted. In any case, if we dropped it, it would follow for the categories we are interested in from our condition (vi) below.

Using the $*$-operation we can define notions of projection, unitary, partial isometry, etc., for morphisms. For example, a morphism $u \in \mathrm{Mor}(U,V)$ is called unitary, if $u^*u = 1$ and $uu^* = 1$.

The category $\mathscr{C}$ is called a C*-tensor category, or a monoidal C*-category, if in addition we are given a bilinear bifunctor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, $(U,V) \mapsto U \otimes V$, natural unitary isomorphisms

$$\alpha_{U,V,W}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$

called the associativity morphisms, an object $\mathbb{1}$, called the unit object, and natural unitary isomorphisms

$$\lambda_U\colon \mathbb{1} \otimes U \to U, \quad \rho_U\colon U \otimes \mathbb{1} \to U,$$

such that

(iii) the pentagonal diagram

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "\alpha\otimes\iota"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(U \otimes (V \otimes W)) \otimes X \arrow[d, "\alpha_{1,23,4}"'] & & (U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{1,2,34}"] \\
U \otimes ((V \otimes W) \otimes X) \arrow[rr, "\iota\otimes\alpha"] & & U \otimes (V \otimes (W \otimes X))
\end{tikzcd}$$

commutes; here the leg-numbering notation for the associativity morphisms means that $\alpha_{12,3,4} = \alpha_{U\otimes V,W,X}$, etc.;

(iv) $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$, and the triangle diagram

$$\begin{tikzcd}
(U \otimes \mathbb{1}) \otimes V \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes\iota"'] & & U \otimes (\mathbb{1} \otimes V) \arrow[dl, "\iota\otimes\lambda"] \\
 & U \otimes V &
\end{tikzcd}$$

commutes;

(v) $(S \otimes T)^* = S^* \otimes T^*$ for any morphisms S and T.

In addition we will also always assume that

(vi) the category $\mathscr{C}$ has finite direct sums, meaning that for any objects U and V there exist an object W and isometries $u \in \mathrm{Mor}(U, W)$ and $v \in \mathrm{Mor}(V, W)$ such that $uu^* + vv^* = 1$;

(vii) the category $\mathscr{C}$ has subobjects, meaning that for every projection $p \in \mathrm{End}(U)$ there exists an object V and an isometry $v \in \mathrm{Mor}(V, U)$ such that $vv^* = p$; note that an object defined by the zero projection $0 \in \mathrm{End}(U)$ for some U is a zero object, that is, an object $\mathbf{0}$ such that $\mathrm{Mor}(\mathbf{0}, W) = 0$ and $\mathrm{Mor}(W, \mathbf{0}) = 0$ for any W;

(viii) the unit object $\mathbb{1}$ is simple, that is, $\mathrm{End}(\mathbb{1}) = \mathbb{C}1 \cong \mathbb{C}$;

(ix) the category is small, that is, the class of objects is a set.

The category $\mathscr{C}$ is called strict if $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, $\mathbb{1} \otimes U = U \otimes \mathbb{1} = U$, and α, λ and ρ are the identity morphisms.

The primary example of a C*-tensor category is the category Hilb_f of finite dimensional Hilbert spaces. To be pedantic, in order to make this category small we need to consider a set of Hilbert spaces instead of all possible spaces. We will assume that such a set is fixed once for all, and it is big enough to accommodate all the constructions we will encounter. We should also fix exactly what realization of tensor products we are using, as well as choose a one-dimensional Hilbert space playing the role of the unit object.

The associativity morphisms in Hilb_f are of course $(\xi \otimes \zeta) \otimes \eta \mapsto \xi \otimes (\zeta \otimes \eta)$. They are so obvious that it is common to consider Hilb_f as a strict tensor category. In fact, it is indeed strict in the rigorous sense with a suitable definition of tensor products. In order to see this, consider the Cuntz algebra $\mathscr{O}_\infty$, the universal unital C*-algebra generated by isometries S_n, $n \geq 1$, such that the projections $S_n S_n^*$ are mutually orthogonal, so that $S_i^* S_j = \delta_{ij} 1$. For every finite dimensional Hilbert space H fix a linear map $\beta_H \colon H \to \mathscr{O}_\infty$ such that $\beta_H(\zeta)^* \beta_H(\xi) = (\xi, \zeta)1$. If H is already a subspace of $\mathscr{O}_\infty$ such that $\zeta^*\xi = (\xi, \zeta)1$ for all $\xi, \zeta \in H$, we take β_H to be the inclusion map. For arbitrary H and K we then define $H \otimes K$ as the subspace of $\mathscr{O}_\infty$ spanned by the vectors $\beta_H(\xi)\beta_K(\zeta)$, and let $\xi \otimes \zeta = \beta_H(\xi)\beta_K(\zeta)$. The unit object is of course $\mathbb{C}1 \subset \mathscr{O}_\infty$.

Independently of how we define tensor products to get the strict tensor C*-category Hilb_f, the unit object is a uniquely defined one-dimensional Hilbert space. It has a unique unit vector ξ such that $\xi \otimes \xi = \xi$. Therefore $\mathbb{1}$ is canonically isomorphic to $\mathbb{C}$. By slightly abusing notation we will henceforth identify $\mathbb{1}$ with $\mathbb{C}$.

Another example of a strict C*-tensor category, which is our main object of interest, is the category $\mathrm{Rep}\, G$ of finite dimensional unitary representations of a compact quantum group G. Again, to be precise, we should assume that the underlying spaces of our representations are elements of the set of Hilbert spaces used to define Hilb_f.

By a result of Mac Lane [**48**, Theorem X1.5.3] (proved though in a slightly different context than we are dealing with now), any tensor category can be strictified. This means that it is equivalent, in a sense that will be explained shortly, to a strict tensor category. It follows that in developing the general theory we may assume that our categories are strict. Mac Lane's result is also useful in formal computations, as it allows one to perform such computations as if the category one works with were strict. This will be used in Section 4.2. At the same time a strictification is not always desirable, since certain categories are better described as nonstrict categories. Let us give a simple example, another important example of the same type, but significantly more complicated, will be given in Section 4.1.

Example 2.1.2. — Let Γ be a discrete group, and $\omega \in Z^3(\Gamma;\mathbb{T})$ be a normalized $\mathbb{T}$-valued 3-cocycle; recall that being normalized means that $\omega(g,h,k) = 1$ whenever one of the elements g, h or k is the unit element. Define a C*-tensor category $\mathscr{C}(\Gamma,\omega)$ as follows. First consider the category $\mathscr{C}(\Gamma) = \operatorname{Rep}\hat\Gamma$ of finite dimensional unitary representations of $\hat\Gamma$. Then define $\mathscr{C}(\Gamma,\omega)$ as the same category $\mathscr{C}(\Gamma)$, except that the associativity morphisms are given by the action of $\omega \in \mathscr{U}(\hat\Gamma^3)$.

More concretely, by Example 1.6.3 we can choose representatives V_g of simple objects of $\mathscr{C}(\Gamma,\omega)$ indexed by elements $g\in\Gamma$. Note that $V_g\otimes V_h \cong V_{gh}$, and we can even arrange $V_g\otimes V_h = V_{gh}$ by taking the spaces H_{V_g} to be the unit $\mathbb{1}$ in Hilb_f. The associativity morphisms are defined by the operators $H_{V_g}\otimes H_{V_h}\otimes H_{V_k} \to H_{V_g}\otimes H_{V_h}\otimes H_{V_k}$ of multiplication by $\omega(g,h,k)$, or in other words, by the operators $\omega(g,h,k)1 \in \operatorname{End}(V_{ghk})$. The commutativity of the pentagon diagram is equivalent to the cocycle identity

$$\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k) = \omega(gh,k,l)\omega(g,h,kl). \qquad \square$$

Definition 2.1.3. — Let $\mathscr{C}$ and $\mathscr{C}'$ be C*-tensor categories. A tensor functor $\mathscr{C}\to\mathscr{C}'$ is a functor $F\colon \mathscr{C}\to\mathscr{C}'$ that is linear on morphisms, together with an isomorphism $F_0\colon \mathbb{1}'\to F(\mathbb{1})$ in $\mathscr{C}'$ and natural isomorphisms

$$F_2\colon F(U)\otimes F(V)\to F(U\otimes V)$$

such that the diagram

$$\begin{tikzcd}
(F(U)\otimes F(V))\otimes F(W) \arrow[r, "F_2\otimes\iota"] \arrow[d, "\alpha'"'] & F(U\otimes V)\otimes F(W) \arrow[r, "F_2"] & F((U\otimes V)\otimes W) \arrow[d, "F(\alpha)"] \\
F(U)\otimes(F(V)\otimes F(W)) \arrow[r, "\iota\otimes F_2"] & F(U)\otimes F(V\otimes W) \arrow[r, "F_2"] & F(U\otimes(V\otimes W))
\end{tikzcd}$$

and the diagrams

$$\begin{tikzcd} F(\mathbb{1})\otimes F(U) \arrow[r, "F_2"] & F(\mathbb{1}\otimes U) \arrow[d, "F(\lambda)"] \\ \mathbb{1}'\otimes F(U) \arrow[u, "F_0\otimes\iota"] \arrow[r, "\lambda'"] & F(U) \end{tikzcd} \qquad \begin{tikzcd} F(U)\otimes F(\mathbb{1}) \arrow[r, "F_2"] & F(U\otimes \mathbb{1}) \arrow[d, "F(\rho)"] \\ F(U)\otimes \mathbb{1}' \arrow[u, "\iota\otimes F_0"] \arrow[r, "\rho'"] & F(U) \end{tikzcd}$$

commute.

We say that the tensor functor is unitary if in addition $F(T)^* = F(T^*)$ on morphisms, and $F_2\colon F(U)\otimes F(U)\to F(U\otimes V)$ and F_0 are unitary.

It can be shown that F_0 is uniquely determined by F and F_2. Furthermore, instead of the existence of F_0 it is enough to assume that $F(\mathbb{1})\cong\mathbb{1}'$. In any case, there will always be an obvious choice for F_0, so we will often omit F_0 in our considerations.

Example 2.1.4. — Let G be a compact quantum group. Define a tensor functor $F\colon \operatorname{Rep} G\to \mathrm{Hilb}_f$ by letting $F(U)=H_U$ for every finite dimensional unitary representation of G, while F_2, as well as the action of F on morphisms, are taken to be the identity maps. □

Definition 2.1.5. — A natural isomorphism $\eta\colon F\to G$ between two tensor functors $\mathscr{C}\to\mathscr{C}'$ is called monoidal if the diagrams

$$\begin{tikzcd} F(U)\otimes F(V) \arrow[r, "F_2"] \arrow[d, "\eta\otimes\eta"'] & F(U\otimes V) \arrow[d, "\eta"] \\ G(U)\otimes G(V) \arrow[r, "G_2"] & G(U\otimes V) \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{1}' \arrow[dl, "F_0"'] \arrow[dr, "G_0"] & \\ F(\mathbb{1}) \arrow[rr, "\eta"] & & G(\mathbb{1}) \end{tikzcd}$$

commute.

Definition 2.1.6. — Two C*-tensor $\mathscr{C}$ and $\mathscr{C}'$ categories are called monoidally equivalent if there exist tensor functors $F\colon\mathscr{C}\to\mathscr{C}'$ and $G\colon\mathscr{C}'\to\mathscr{C}$ such that FG and GF are naturally monoidally isomorphic to the identity functors. If we can choose F, G and the natural isomorphisms $FG\cong\iota$ and $GF\cong\iota$ to be unitary, then we say that $\mathscr{C}$ and $\mathscr{C}'$ are unitarily monoidally equivalent.

In practice we will use the following criterion of equivalence rather than the above definition: a (unitary) tensor functor $F\colon\mathscr{C}\to\mathscr{C}'$ is a (unitary) monoidal equivalence if and only if F is fully faithful (that is, it defines an isomorphism between $\operatorname{Mor}(U,V)$ and $\operatorname{Mor}(F(U),F(V))$) and essentially surjective (that is, every object in $\mathscr{C}'$ is isomorphic to $F(U)$ for some U). Furthermore, we will mainly deal with semisimple categories, meaning that every object is a direct sum of simple ones. For such a category $\mathscr{C}$ choose representatives U_α of the isomorphism classes of simple objects. Then a tensor functor $F\colon\mathscr{C}\to\mathscr{C}'$, where $\mathscr{C}'$ is another semisimple C*-tensor category, is a monoidal equivalence if and only if the objects $F(U_\alpha)$ are simple, pairwise nonisomorphic, and any simple object in $\mathscr{C}'$ is isomorphic to some $F(U_\alpha)$.

Example 2.1.7. — Let Γ be a discrete group, ω and ω' be normalized $\mathbb{T}$-valued 3-cocycles on Γ. Consider the categories $\mathscr{C}(\Gamma,\omega)$ and $\mathscr{C}(\Gamma,\omega')$ defined in Example 2.1.2. Assume c is a $\mathbb{T}$-valued 2-cochain on Γ such that $\omega'\omega^{-1} = \partial c$, so

$$(\omega'\omega^{-1})(g,h,k) = c(h,k)c(gh,k)^{-1}c(g,hk)c(g,h)^{-1}.$$

Then we can define a tensor functor $F\colon \mathscr{C}(\Gamma,\omega) \to \mathscr{C}(\Gamma,\omega')$ that is the identity map on objects and morphisms, while the tensor structure F_2 is given by the action of $c^{-1} \in \mathscr{U}(\hat{\Gamma}^2)$. Therefore the categories $\mathscr{C}(\Gamma,\omega)$ and $\mathscr{C}(\Gamma,\omega')$ are unitarily monoidally equivalent. So up to equivalence $\mathscr{C}(\Gamma,\omega)$ depends only on the cohomology class of ω. Furthermore, it is not difficult to see that $\mathscr{C}(\Gamma,\omega)$ and $\mathscr{C}(\Gamma,\omega')$ are unitarily monoidally equivalent if and only if there exists an automorphism β of Γ such that the cocycles $\beta(\omega)$ and ω' are cohomologous. $\square$

As we already mentioned, but now can be more precise, a theorem of Mac Lane asserts that any C*-tensor category is unitarily monoidally equivalent to a strict C*-tensor category. In view of Example 2.1.7 this might seem counterintuitive, since it is natural to assume that the cohomology class of ω is an obstruction for strictification of $\mathscr{C}(\Gamma,\omega)$. Note, however, that since free groups have cohomological dimension one, any 3-cocycle on Γ becomes a coboundary when lifted to the free group with generators in Γ. The proof of Mac Lane's theorem is based on a similar idea: the new strict category equivalent to $\mathscr{C}$ has objects that are n-tuples of objects in $\mathscr{C}$, while the tensor product is defined by concatenation, see the proof of [**48**, Theorem XI.5.3] for details.

References. — [**25**], [**45**], [**48**], [**61**].

2.2. CONJUGATE OBJECTS AND INTRINSIC DIMENSION

Let $\mathscr{C}$ be a strict C*-tensor category.

Definition 2.2.1. — An object $\bar{U}$ is said to be conjugate to an object U in $\mathscr{C}$ if there exist morphisms $R\colon \mathbb{1} \to \bar{U}\otimes U$ and $\bar{R}\colon \mathbb{1} \to U\otimes\bar{U}$ such that

$$U \xrightarrow{\iota\otimes R} U\otimes\bar{U}\otimes U \xrightarrow{\bar{R}^*\otimes\iota} U \quad\text{and}\quad \bar{U} \xrightarrow{\iota\otimes\bar{R}} \bar{U}\otimes U\otimes\bar{U} \xrightarrow{R^*\otimes\iota} \bar{U}$$

are the identity morphisms. The identities $(\bar{R}^*\otimes\iota)(\iota\otimes R) = \iota$ and $(R^*\otimes\iota)(\iota\otimes\bar{R}) = \iota$ are called the conjugate equations.

If every object has a conjugate object, then $\mathscr{C}$ is said to be a C*-tensor category with conjugates, or a rigid C*-tensor category.

Note that this definition is symmetric in U and $\bar{U}$, so U is conjugate to $\bar{U}$.

The notion of a conjugate object can of course be defined also for nonstrict categories. Then the conjugate equations involve the associativity morphisms and the isomorphisms λ and ρ. But since any tensor category can be strictified, in developing the theory it suffices to consider strict tensor categories.

Example 2.2.2. — Consider $\mathscr{C} = \mathrm{Hilb}_f$. If H is a finite dimensional Hilbert space with an orthonormal basis $\{e_i\}_i$, define

$$r\colon \mathbb{C} \to \bar{H} \otimes H \ \text{ by } \ r(1) = \sum_i \bar{e}_i \otimes e_i,$$

and

$$\bar{r}\colon \mathbb{C} \to H \otimes \bar{H} \ \text{ by } \ \bar{r}(1) = \sum_i e_i \otimes \bar{e}_i.$$

Then the pair $(r, \bar{r})$ solves the conjugate equations for H. Hence $\bar{H}$ is a conjugate object to H. Note that r and $\bar{r}$ do not depend on the choice of an orthonormal basis. At the same time the pair $(r, \bar{r})$ is not the only possible: for any invertible operator $T \in B(H)$ the maps

$$(1 \otimes T^*)r = (j(T)^* \otimes 1)r \ \text{ and } \ (T^{-1} \otimes 1)\bar{r} = (1 \otimes j(T)^{-1})\bar{r}$$

also solve the conjugate equations; recall that j denotes the canonical anti-isomorphism of $B(H)$ onto $B(\bar{H})$, $j(T)\bar{\xi} = \overline{T^*\xi}$ for $\xi \in H$. □

Example 2.2.3. — Consider $\mathscr{C} = \operatorname{Rep} G$, where G is a compact quantum group. Let $U \in B(H) \otimes C(G)$ be a finite dimensional unitary representation. Consider the maps r and $\bar{r}$ from the previous example. A straightforward computation shows that $\bar{r} \in \operatorname{Mor}(\mathbb{1}, U \times U^c)$ (more generally, if we identify $H \otimes \bar{H}$ with $B(H)$, then the map $\operatorname{Mor}(\mathbb{1}, U \times U^c) \to B(H)$, $f \mapsto f(1)$, defines an isomorphism of $\operatorname{Mor}(\mathbb{1}, U \times U^c)$ onto $\operatorname{End}(U)$ that maps $\bar{r}$ into $1 \in \operatorname{End}(U)$). Consider the operator $\rho_U \in \operatorname{Mor}(U, U^{cc})$ introduced in Section 1.4. By the definition of the conjugate representation we have $j(\rho_U)^{1/2} \in \operatorname{Mor}(U^c, \bar{U})$. Hence the operator

$$\bar{R} = (1 \otimes j(\rho_U)^{1/2})\bar{r} = (\rho_U^{1/2} \otimes 1)\bar{r}$$

belongs to $\operatorname{Mor}(\mathbb{1}, U \times \bar{U})$. Replacing U by $\bar{U}$ we also get an operator

$$R = (1 \otimes j(\rho_{\bar{U}})^{1/2})r \in \operatorname{Mor}(\mathbb{1}, \bar{U} \times U).$$

By Proposition 1.4.7 we have $\rho_{\bar{U}} = j(\rho_U)^{-1}$, hence

$$R = (1 \otimes \rho_U^{-1/2})r.$$

Therefore $(R, \bar{R})$ is a solution of the conjugate equations for U, so $\bar{U}$ is a conjugate object to U. □

Example 2.2.4. — Consider the category $\mathscr{C}(\Gamma,\omega)$ defined in Example 2.1.2. Unless the cocycle ω is trivial, this is a nonstrict category, but as we mentioned above, the notion of a conjugate object still makes sense. Then as a conjugate to V_g we can take $V_{g^{-1}}$, with $R\colon V_e\to V_{g^{-1}}\otimes V_g=V_e$ to be the identity map and $\bar R\colon V_e\to V_g\otimes V_{g^{-1}}=V_e$ the multiplication by $\overline{\omega(g,g^{-1},g)}=\omega(g^{-1},g,g^{-1})$ (this equality follows by applying the cocycle identity to (g,g^{-1},g,g^{-1})). □

Proposition 2.2.5. — *For any object U in $\mathscr{C}$ a conjugate object, if it exists, is uniquely determined up to an isomorphism. More precisely, if $(R,\bar R)$ is a solution of the conjugate equations for U and $\bar U$, and $(R',\bar R')$ is a solution of the conjugate equations for U and $\bar U'$, then*

$$T=(\iota_{\bar U}\otimes\bar R'^{*})(R\otimes\iota_{\bar U'})\in\mathrm{Mor}(\bar U',\bar U)$$

is invertible with inverse $S=(\iota_{\bar U'}\otimes\bar R^{*})(R'\otimes\iota_{\bar U})$, and

$$R'=(T^{-1}\otimes\iota)R,\ \ \bar R'=(\iota\otimes T^{*})\bar R.$$

Proof. — We compute:

$$\begin{aligned}TS&=(\iota\otimes\bar R'^{*})(R\otimes\iota)(\iota\otimes\bar R^{*})(R'\otimes\iota)\\&=(\iota\otimes\bar R'^{*})(\iota\otimes\iota\otimes\iota\otimes\bar R^{*})(R\otimes\iota\otimes\iota\otimes\iota)(R'\otimes\iota)\\&=(\iota\otimes\bar R^{*})(\iota\otimes\bar R'^{*}\otimes\iota\otimes\iota)(\iota\otimes\iota\otimes R'\otimes\iota)(R\otimes\iota)\\&=(\iota\otimes\bar R^{*})(R\otimes\iota)=\iota.\end{aligned}$$

Similarly one checks that $ST=\iota$. Next,

$$(T^{-1}\otimes\iota)R=(\iota\otimes\bar R^{*}\otimes\iota)(R'\otimes\iota\otimes\iota)R=(\iota\otimes\bar R^{*}\otimes\iota)(\iota\otimes\iota\otimes R)R'=R'.$$

Similarly, $(\iota\otimes T^{*})\bar R=\bar R'$. □

We also have the following related important result, proved by a straightforward computation.

***Theorem 2.2.6* (Frobenius reciprocity).** — *If an object U has a conjugate, with R and $\bar R$ solving the conjugate equations, then the map*

$$\mathrm{Mor}(U\otimes V,W)\to\mathrm{Mor}(V,\bar U\otimes W),\ \ T\mapsto(\iota_{\bar U}\otimes T)(R\otimes\iota_V),$$

is a linear isomorphism, and the inverse map is given by $S\mapsto(\bar R^{*}\otimes\iota_W)(\iota_U\otimes S)$. Similarly, $\mathrm{Mor}(V\otimes U,W)\simeq\mathrm{Mor}(V,W\otimes\bar U)$.

Corollary 2.2.7. — *Assume that U is simple and that $\bar U$ is conjugate to U. Then $\bar U$ is simple and the spaces $\mathrm{Mor}(\mathbb{1},\bar U\otimes U)$ and $\mathrm{Mor}(\mathbb{1},U\otimes\bar U)$ are one-dimensional.*

Proof. — By the Frobenius reciprocity the spaces $\mathrm{End}(\bar U)$, $\mathrm{Mor}(\mathbb{1},\bar U\otimes U)$ and $\mathrm{Mor}(\mathbb{1},U\otimes\bar U)$ are isomorphic to $\mathrm{End}(U)=\mathbb{C}1$. □

Proposition 2.2.8. — *Assume that an object U has a conjugate. Then* $\mathrm{End}(U)$ *is finite dimensional.*

Proof. — We will prove the proposition by showing that there exists a positive linear functional f on $\mathrm{End}(U)$ such that $T \le f(T)1$ for all $T \ge 0$, which is possible only for finite dimensional C*-algebras.

In order to do this, define $\varphi_U\colon \mathrm{End}(U) \to \mathrm{End}(\mathbb{1})$ by letting $\varphi_U(T)$ to be equal to the composition

$$\mathbb{1} \xrightarrow{R} \bar U \otimes U \xrightarrow{\iota\otimes T} \bar U \otimes U \xrightarrow{R^*} \mathbb{1}.$$

For $X \in \mathrm{End}(U)$ consider $Y = (\iota \otimes X)R \in \mathrm{Mor}(\mathbb{1}, \bar U \times U)$. Then

$$X = (\bar R^* \otimes \iota_U)(\iota_U \otimes Y) \in \mathrm{Mor}(U \otimes \mathbb{1}, \mathbb{1} \otimes U) = \mathrm{End}(U).$$

It follows that

$$X^*X \le \|\bar R\|^2(\iota \otimes Y^*Y) = \|\bar R\|^2(\iota \otimes \varphi_U(X^*X))$$

Therefore if the functional f is defined by $\|\bar R\|^2\varphi_U(T) = f(T)1$, then $f(T)1 \ge T$ for all $T \in \mathrm{End}(U)_+$. □

Corollary 2.2.9. — *Every object with a conjugate decomposes into a finite direct sum of simple objects.*

Proposition 2.2.10. — *The class of objects in $\mathscr{C}$ that have conjugates forms a C*-tensor subcategory of $\mathscr{C}$.*

Proof. — It is easy to see that the class of objects with conjugates is closed under taking direct sums. We have to show that it is also closed under taking tensor products and subobjects.

Assume $(R_U, \bar R_U)$ is a solution of the conjugate equations for U, and $(R_V, \bar R_V)$ is a solution of the conjugate equations for V. Viewing R_V as a morphism $\mathbb{1} \to \bar V \otimes \mathbb{1} \otimes V$ put

$$R = (\iota \otimes R_U \otimes \iota)R_V \in \mathrm{Mor}(\mathbb{1}, \bar V \otimes \bar U \otimes U \otimes V).$$

Similarly define $\bar R = (\iota \otimes \bar R_V \otimes \iota)\bar R_U \in \mathrm{Mor}(\mathbb{1}, U \otimes V \otimes \bar V \otimes \bar U)$. Then $(R, \bar R)$ is a solution of the conjugation equations for $U \otimes V$, so $\bar V \otimes \bar U$ is conjugate to $U \otimes V$.

Assume now that $(R_U, \bar R_U)$ is a solution of the conjugate equations for U, and that V is subobject of U, so there exists an isometry $w \in \mathrm{Mor}(V, U)$. By the Frobenius reciprocity, there exists a linear isomorphism $\mathrm{End}(U) \to \mathrm{End}(\bar U)$, $T \mapsto T^\vee$, uniquely defined by the identities

$$(\iota \otimes T)R_U = (T^\vee \otimes \iota)R_U.$$

This linear isomorphism is anti-multiplicative (but not, in general, $*$-preserving), as

$$(\iota \otimes ST)R_U = (T^\vee \otimes S)R_U = (T^\vee S^\vee \otimes \iota)R_U.$$

Hence $e = (ww^*)^\vee \in \mathrm{End}(\bar U)$ is an idempotent. Then e is equivalent to a projection, and since $\mathscr{C}$ is assumed to have subobjects, there exists an object $\bar V$ and morphisms $S \in \mathrm{Mor}(\bar V, \bar U)$ and $T \in \mathrm{Mor}(\bar U, \bar V)$ such that $TS = \iota_{\bar V}$ and $ST = e$. Put

$$R_V = (T \otimes w^*)R_U \quad\text{and}\quad \bar R_V = (w^* \otimes S^*)\bar R_U.$$

Then $(R_V, \bar R_V)$ is a solution of the conjugate equations for V. Let us, for example, check that $(\bar R_V^* \otimes \iota)(\iota \otimes R_V) = \iota_V$:

$$\begin{aligned}(\bar R_V^* \otimes \iota)(\iota \otimes R_V) &= (\bar R_U^* \otimes \iota)(w \otimes ST \otimes w^*)(\iota \otimes R_U)\\ &= w^*(\bar R_U^* \otimes \iota)(\iota \otimes e \otimes \iota)(\iota \otimes R_U)w\\ &= w^*(\bar R_U^* \otimes \iota)(\iota \otimes \iota \otimes ww^*)(\iota \otimes R_U)w\\ &= w^*ww^*w = \iota_V.\end{aligned}$$ □

Assume now till the end of the section that $\mathscr{C}$ is a C*-tensor category with conjugates.

Let U be a simple object, $\bar U$ be conjugate to U, and $(R, \bar R)$ be a solution of the conjugate equations for U and $\bar U$. By Proposition 2.2.5 or by Corollary 2.2.7 any other solution of the conjugate equations for U and $\bar U$ has the form $R' = \bar\lambda R$, $\bar R' = \lambda^{-1}\bar R$ for some $\lambda \in \mathbb{C}^*$. In particular, the number $\|R\| \cdot \|\bar R\|$ is independent of the solution.

Definition 2.2.11. — The number

$$d_i(U) = \|R\| \cdot \|\bar R\|$$

is called the intrinsic dimension of the simple object U.

For general U, decompose U into a direct sum of simple objects, $U = \oplus_k U_k$, and put $d_i(U) = \sum_k d_i(U_k)$.

Note that we always have $d_i(\mathbb{1}) = 1$.

Example 2.2.12. — For $\mathscr{C} = \mathrm{Hilb}_f$ we have $d_i(U) = \dim U$, since every object is a direct sum of copies of the unit object. We get the same identity $d_i(U) = \dim U$ for the categories $\mathscr{C}(\Gamma, \omega)$, since $d_i(V_g) = 1$ by Example 2.2.4. □

Example 2.2.13. — Consider $\mathscr{C} = \mathrm{Rep}\, G$, where G is a compact quantum group. Let U be an irreducible unitary representation of G. In Example 2.2.3 we showed that the operators $R = (1 \otimes \rho_U^{-1/2})r$ and $\bar R = (\rho_U^{1/2} \otimes 1)\bar r$ solve the conjugate equations. Let $\{e_i\}_i$ be an orthonormal basis in H_U. Then

$$\|R\| = \|(1 \otimes \rho_U^{-1/2})r(1)\| = \Big\|\sum_i \bar e_i \otimes \rho_U^{-1/2} e_i\Big\| = \mathrm{Tr}(\rho_U^{-1})^{1/2} = (\dim_q U)^{1/2}.$$

Similarly, $\|\bar R\| = (\dim_q U)^{1/2}$. Therefore $d_i(U) = \dim_q U$. □

The intrinsic dimension $d_i(U)$ for nonsimple objects can be expressed in terms of solutions of the conjugate equations. In order to see this, let us introduce a particular class of such solutions. Decompose U into a direct sum of simple objects: $U = \oplus_k U_k$. This decomposition means that we fix isometries $w_k \in \mathrm{Mor}(U_k, U)$ such that $\sum_k w_k w_k^* = 1$. For every k choose a conjugate $\bar U_k$ to U_k. Let $\bar U$ be the direct sum of $\bar U_k$, and let $\bar w_k \in \mathrm{Mor}(\bar U_k, \bar U)$ be the corresponding isometries. Let $(R_k, \bar R_k)$ be a solution of the conjugate equations for U_k and $\bar U_k$. Then $R = \sum_k (\bar w_k \otimes w_k) R_k$, $\bar R = \sum_k (w_k \otimes \bar w_k) \bar R_k$ is a solution of the conjugate equations for U and $\bar U$.

Definition 2.2.14. — A solution of the conjugate equations for U and $\bar U$ of the form

$$R = \sum_k (\bar w_k \otimes w_k) R_k, \quad \bar R = \sum_k (w_k \otimes \bar w_k) \bar R_k,$$

with U_k simple and $\|R_k\| = \|\bar R_k\| = d_i(U_k)^{1/2}$ for all k, is called standard.

For standard solutions we have the following refinement of Proposition 2.2.5.

Proposition 2.2.15. — *Assume $(R, \bar R)$ is a standard solution of the conjugate equations for U and $\bar U$, and $(R', \bar R')$ is another standard solution of the conjugate equations for U and $\bar U'$. Then there exists a unitary $T \in \mathrm{Mor}(\bar U, \bar U')$ such that $R' = (T \otimes \iota) R$ and $\bar R' = (\iota \otimes T) \bar R$.*

Proof. — In order to simplify the notation we will only consider the case when U is a direct sum of n objects isomorphic to one simple object V. Then we can write

$$R = \sum_k (\bar w_k \otimes w_k) R_0, \quad \bar R = \sum_k (w_k \otimes \bar w_k) \bar R_0,$$

and similarly for R' and $\bar R'$ using isometries w'_j and $\bar w'_j$, where $(R_0, \bar R_0)$ is a fixed solution of the conjugate equations for V with $\|R_0\| = \|\bar R_0\|$. The projections $w_k w_k^*$, as well as the projections $w'_j w'^*_j$, are minimal projections in $\mathrm{End}(U) \cong \mathrm{Mat}_n(\mathbb{C})$ that add up to one. Hence there exists a unitary $S \in \mathrm{End}(U)$ such that $S w_k w_k^* S^* = w'_k w'^*_k$. The isometry $S w_k$ must coincide with w'_k up to a phase factor, hence by changing S we can arrange so that $S w_k = w'_k$. Let s_{kj} be the scalar $w_k^* S w_j \in \mathrm{End}(V) \cong \mathbb{C}$. Then $(s_{kj})_{k,j}$ is a unitary matrix, and if we let $T = \sum_{j,k} s_{kj} \bar w'_j \bar w_k^*$, then T is unitary,

$$w'_j = S w_j = \sum_k s_{kj} w_k \quad \text{and} \quad T \bar w_k = \sum_j s_{kj} \bar w'_j.$$

It follows that

$$\sum_k T \bar w_k \otimes w_k = \sum_{k,j} s_{kj} \bar w'_j \otimes w_k = \sum_j \bar w'_j \otimes w'_j,$$

so $R' = (T \otimes \iota) R$. For the same reason $\bar R' = (\iota \otimes T) \bar R$. □

If $R = \sum_k (\bar w_k \otimes w_k) R_k$ and $\bar R = \sum_k (w_k \otimes \bar w_k) \bar R_k$ form a standard solution, then

$$\|R\|^2 1 = R^* R = \sum_k R_k^* R_k = \sum_k d_i(U_k) 1 = d_i(U) 1.$$

Similarly $\|\bar{R}\| = d_i(U)^{1/2}$. To make this useful, we need an intrinsic characterization of standard solutions.

Theorem 2.2.16. — *Let $(R, \bar{R})$ be a solution of the conjugate equations for U and $\bar{U}$. Define a map $\varphi_U\colon \operatorname{End}(U) \to \operatorname{End}(\mathbb{1})$ by letting $\varphi_U(T)$ to be equal to the composition*

$$\mathbb{1} \xrightarrow{R} \bar{U} \otimes U \xrightarrow{\iota \otimes T} \bar{U} \otimes U \xrightarrow{R^*} \mathbb{1},$$

and similarly define $\psi_U\colon \operatorname{End}(U) \to \operatorname{End}(\mathbb{1})$ by letting $\psi_U(T)$ to be equal to the composition

$$\mathbb{1} \xrightarrow{\bar{R}} U \otimes \bar{U} \xrightarrow{T \otimes \iota} U \otimes \bar{U} \xrightarrow{\bar{R}^*} \mathbb{1}.$$

Then $(R, \bar{R})$ is standard if and only if $\varphi_U = \psi_U$. Furthermore, for the standard solutions the maps φ_U and ψ_U are tracial, positive and faithful, and they do not depend on the choice of a standard solution.

Proof. — If $(R, \bar{R})$ is a standard solution, so

$$R = \sum_k (\bar{w}_k \otimes w_k) R_k, \quad \bar{R} = \sum_k (w_k \otimes \bar{w}_k) \bar{R}_k,$$

then

$$\varphi_U(T) = \sum_k \varphi_{U_k}(w_k^* T w_k) \ \text{ and } \ \psi_U(T) = \sum_k \psi_{U_k}(w_k^* T w_k).$$

We clearly have $\varphi_{U_k} = \psi_{U_k}$, since both maps send $1 \in \operatorname{End}(U_k) = \mathbb{C}1$ into $d_i(U_k)1 = \|R_k\|^2 1 = \|\bar{R}_k\|^2 1$. Hence $\varphi_U = \psi_U$.

We will next check that φ_U is tracial. In order to simplify the notation assume, as in the proof of the previous proposition, that all the objects U_k coincide with one simple object V. In this case for any $T \in \operatorname{End}(U)$ the morphism $w_k^* T w_l \in \operatorname{End}(V)$ is a scalar, and we can identify the C*-algebra $\operatorname{End}(U)$ with $\operatorname{Mat}_n(\mathbb{C})$ via the map $T \mapsto (w_k^* T w_l)_{k,l}$. Then $\varphi_U(T) = \operatorname{Tr}(T)\varphi_V(1)$. Therefore φ_U is tracial, positive and faithful.

The fact that φ_U and ψ_U are independent of the choice of a standard solution is an immediate consequence of Proposition 2.2.15.

Finally, assume that $(R', \bar{R}')$ is an arbitrary solution of the conjugate equations for the same $\bar{U}$. By Proposition 2.2.5, applied to the pair $(\bar{U}, U)$ rather than to $(U, \bar{U})$, there exists an invertible element $T \in \operatorname{End}(U)$ such that $R' = (\iota \otimes T^*)R$ and $\bar{R}' = (T^{-1} \otimes \iota)\bar{R}$. Then for any $S \in \operatorname{End}(U)$ we have

$$\varphi'_U(S) = \varphi_U(TST^*) = \varphi_U(ST^*T)$$

and

$$\psi'_U(S) = \psi_U((T^*)^{-1} S T^{-1}) = \psi_U(S T^{-1} (T^*)^{-1}) = \psi_U(S(T^*T)^{-1}).$$

It follows that $\varphi'_U = \psi'_U$ if and only if $T^*T = (T^*T)^{-1}$, that is, T is unitary. Then $(R', \bar{R}')$ is clearly standard, defined by the isometries $T^* w_k$ and $\bar{w}_k$. $\square$

Remark 2.2.17. — If $\mathscr{C} = \operatorname{Rep} G$ for a compact quantum group G, then for R and $\bar{R}$ defined in Example 2.2.3 we have $\varphi_U = \operatorname{Tr}(\cdot\, \rho_U^{-1})1$ and $\psi_U = \operatorname{Tr}(\cdot\, \rho_U)1$. So up to a scalar factor the maps φ_U and ψ_U coincide with the states defined in Section 1.4, and the condition $\varphi_U(T) = \psi_U(T)$ for $T \in \operatorname{End}(U)$ is exactly the condition used there to define ρ_U.

The map $\varphi_U = \psi_U \colon \operatorname{End}(U) \to \operatorname{End}(\mathbb{1}) \cong \mathbb{C}$ defined by a standard solution of the conjugate equations is called the trace and denoted by Tr_U. By definition, $d_i(U) = \operatorname{Tr}_U(1)$. Note also that if $S \in \operatorname{Mor}(U, V)$ and $T \in \operatorname{Mor}(V, U)$, then we can extend S and T to the endomorphisms $\tilde{S} = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ of $U \oplus V$, and then get

$$\operatorname{Tr}_V(ST) = \operatorname{Tr}_{U\oplus V}(\tilde{S}\tilde{T}) = \operatorname{Tr}_{U\oplus V}(\tilde{T}\tilde{S}) = \operatorname{Tr}_U(TS).$$

Extending Theorem 1.4.9 we can now prove multiplicativity of the intrinsic dimension on tensor products.

Theorem 2.2.18. — *For any $S \in \operatorname{End}(U)$ and $T \in \operatorname{End}(V)$ we have*

$$\operatorname{Tr}_{U\otimes V}(S \otimes T) = \operatorname{Tr}_U(S)\operatorname{Tr}_V(T).$$

In particular, $d_i(U \otimes V) = d_i(U)\, d_i(V)$.

Proof. — Let $(R_U, \bar{R}_U)$ be a standard solution of the conjugate equations for U, and $(R_V, \bar{R}_V)$ be a standard solution of the conjugate equations for V. Then, as we have already used in the proof of Proposition 2.2.10, we can define a solution of the conjugate equations for $U \otimes V$ by

$$R = (\iota \otimes R_U \otimes \iota)R_V \quad \text{and} \quad \bar{R} = (\iota \otimes \bar{R}_V \otimes \iota)\bar{R}_U.$$

We claim that this solution is standard. In order to see this define the maps $\varphi_{U\otimes V}, \psi_{U\otimes V} \colon \operatorname{End}(U \otimes V) \to \operatorname{End}(\mathbb{1})$ as in Theorem 2.2.16 using $(R, \bar{R})$. For $Q \in \operatorname{End}(U \otimes V)$ we compute:

$$\begin{aligned}
\varphi_{U\otimes V}(Q) &= R_V^*(\iota \otimes R_U^* \otimes \iota)(\iota \otimes \iota \otimes Q)(\iota \otimes R_U \otimes \iota)R_V \\
&= \varphi_V((R_U^* \otimes \iota)(\iota \otimes Q)(R_U \otimes \iota)) \\
&= \psi_V((R_U^* \otimes \iota)(\iota \otimes Q)(R_U \otimes \iota)) \\
&= \bar{R}_V^*(R_U^* \otimes \iota \otimes \iota)(\iota \otimes Q \otimes \iota)(R_U \otimes \iota \otimes \iota)\bar{R}_V \\
&= (R_U^* \otimes \bar{R}_V^*)(\iota \otimes Q \otimes \iota)(R_U \otimes \bar{R}_V).
\end{aligned}$$

Similarly one checks that

$$\psi_{U\otimes V}(Q) = (R_U^* \otimes \bar{R}_V^*)(\iota \otimes Q \otimes \iota)(R_U \otimes \bar{R}_V).$$

Therefore $\varphi_{U\otimes V} = \psi_{U\otimes V}$, so $(R, \bar{R})$ is standard. Then from the above identities we get

$$\mathrm{Tr}_{U\otimes V}(S\otimes T) = (R_U^* \otimes \bar{R}_V^*)(\iota\otimes S\otimes T\otimes\iota)(R_U\otimes\bar{R}_V) = \mathrm{Tr}_U(S)\,\mathrm{Tr}_V(T). \qquad \square$$

We next give one more characterization of the intrinsic dimension.

Theorem 2.2.19. — *For every object U we have*

$$d_i(U) = \min\{\|\bar{R}\|\cdot\|R\|\},$$

where the minimum is taken over all solutions of the conjugate equations for U. Furthermore, the minimum is attained exactly on the solutions that up to scalar factors coincide with standard solutions.

Proof. — Let $(R, \bar{R})$ be a standard solution. Then we already know that $d_i(U) = \|R\|^2 = \|\bar{R}\|^2$. By Proposition 2.2.5 any other solution for the same $\bar{U}$ has the form $R' = (\iota\otimes T^*)R$, $\bar{R}' = (T^{-1}\otimes\iota)\bar{R}$ for an invertible element $T\in\mathrm{End}(U)$. Then

$$R'^*R' = \mathrm{Tr}_U(TT^*) \ \text{ and } \ \bar{R}'^*\bar{R}' = \mathrm{Tr}_U((T^*)^{-1}T^{-1}) = \mathrm{Tr}_U((TT^*)^{-1}).$$

By the Cauchy-Schwarz inequality we have

$$\mathrm{Tr}_U(1) \le \mathrm{Tr}_U(TT^*)^{1/2}\,\mathrm{Tr}_U((TT^*)^{-1})^{1/2},$$

and the equality holds if and only if TT^* and $(TT^*)^{-1}$ are colinear, that is, TT^* is a scalar, or in other words, T is a unitary multiplied by a scalar. But in this case $(R', \bar{R}')$ coincides modulo scalar factors with a standard solution. $\square$

Corollary 2.2.20. — *Let $\mathscr{C}'$ be a C*-tensor category with conjugates and $F\colon \mathscr{C}\to\mathscr{C}'$ be a unitary tensor functor. Then $d_i(F(U)) \le d_i(U)$ for any object U in $\mathscr{C}$.*

Proof. — Let $(R, \bar{R})$ be a standard solution of the conjugate equations for U. Define $R'\colon \mathbb{1}'\to F(\bar{U})\otimes F(U)$ as the composition

$$\mathbb{1}' \xrightarrow{F_0} F(\mathbb{1}) \xrightarrow{F(R)} F(\bar{U}\otimes U) \xrightarrow{F_2^*} F(\bar{U})\otimes F(U),$$

and similarly define $\bar{R}'\colon \mathbb{1}'\to F(U)\otimes F(\bar{U})$ as the composition

$$\mathbb{1}' \xrightarrow{F_0} F(\mathbb{1}) \xrightarrow{F(\bar{R})} F(U\otimes\bar{U}) \xrightarrow{F_2^*} F(U)\otimes F(\bar{U}).$$

One can check that $(R', \bar{R}')$ is a solution of the conjugate equations for $F(U)$. Furthermore, $\|R'\| = \|R\| = d_i(U)^{1/2}$, since $R'^*R' = F_0^*F(R^*R)F_0 = \|R\|^2 1$, and similarly $\|\bar{R}'\| = d_i(U)^{1/2}$. Hence $d_i(F(U)) \le d_i(U)$. $\square$

We finish the section by showing how the operation of taking a conjugate can be extended to a contravariant functor. Essentially this was already done in the proof of Proposition 2.2.10.

For every object U fix a conjugate object $\bar{U}$ and a solution $(R_U, \bar{R}_U)$ of the conjugate equations. By the Frobenius reciprocity, there exists a linear isomorphism $\mathrm{Mor}(U,V) \to \mathrm{Mor}(\bar{V},\bar{U})$, $T \mapsto T^\vee$, uniquely defined by the identities

$$(\iota \otimes T)R_U = (T^\vee \otimes \iota)R_V.$$

Explicitly,

$$T^\vee = (\iota \otimes \bar{R}_V^*)(\iota \otimes T \otimes \iota)(R_U \otimes \iota).$$

Note that $T^\vee$ is also defined by the identity

$$\bar{R}_V^*(T \otimes \iota) = \bar{R}_U^*(\iota \otimes T^\vee).$$

Theorem 2.2.21. — *The maps $U \mapsto \bar{U}$ and $T \mapsto T^\vee$ define a contravariant functor $\mathscr{C} \to \mathscr{C}$. If all the solutions of the conjugate equations used to define this functor are standard, then the functor is unitary and its square is naturally unitarily isomorphic to the identity functor.*

Proof. — The same computation as in the proof of Proposition 2.2.10 shows that $(ST)^\vee = T^\vee S^\vee$, so we indeed get a contravariant functor.

Assuming now that the solutions of the conjugate equations are standard, in order to check that $T^{*\vee} = T^{\vee *}$ for $T \in \mathrm{Mor}(U,V)$ it suffices to show that $\mathrm{Tr}_{\bar{U}}(ST^{*\vee}) = \mathrm{Tr}_{\bar{U}}(ST^{\vee *})$ for all $S \in \mathrm{Mor}(\bar{V},\bar{U})$. We compute:

$$\begin{aligned}\mathrm{Tr}_{\bar{U}}(ST^{*\vee}) &= R_U^*(ST^{*\vee} \otimes \iota)R_U = R_U^*(S \otimes T^*)R_V = ((\iota \otimes T)R_U)^*(S \otimes \iota)R_V \\ &= R_V^*(T^{\vee *}S \otimes \iota)R_V = \mathrm{Tr}_{\bar{V}}(T^{\vee *}S) = \mathrm{Tr}_{\bar{U}}(ST^{\vee *}).\end{aligned}$$

Hence $T^{*\vee} = T^{\vee *}$.

Since $(\bar{R}_U, R_U)$ and $(R_{\bar{U}}, \bar{R}_{\bar{U}})$ are both standard solutions of the conjugate equations for $\bar{U}$, by Proposition 2.2.15 there exists a unitary $\eta_U \in \mathrm{Mor}(U,\bar{\bar{U}})$ such that

$$(\iota \otimes \eta_U)R_U = \bar{R}_{\bar{U}} \ \text{ and } \ (\eta_U \otimes \iota)\bar{R}_U = R_{\bar{U}}.$$

For $T \in \mathrm{Mor}(U,V)$ we have

$$\begin{aligned}(\iota \otimes \eta_V T)R_U &= (T^\vee \otimes \eta_V)R_V = (T^\vee \otimes \iota)\bar{R}_{\bar{V}} = (\bar{R}_{\bar{V}}^*(T^{\vee *} \otimes \iota))^* \\ &= (\bar{R}_{\bar{U}}^*(\iota \otimes T^{\vee * \vee}))^* = (\iota \otimes T^{\vee * \vee *})\bar{R}_{\bar{U}} = (\iota \otimes T^{\vee * \vee *}\eta_U)R_U,\end{aligned}$$

so $T^{\vee * \vee *}\eta_U = \eta_V T$. Since $T^{*\vee} = T^{\vee *}$, we thus get $T^{\vee\vee}\eta_U = \eta_V T$. Hence the unitaries η_U define a natural isomorphism between the identity functor and the functor $U \mapsto \bar{\bar{U}}$. □

Remark 2.2.22. —

(i) As we see from the proof, for any solutions $(R_U, \bar{R}_U)$ and $(R_V, \bar{R}_V)$ of the conjugate equations for U and V, if we let $\bar{R}_{\bar{U}} = R_U$ and $\bar{R}_{\bar{V}} = R_V$, then $T^{\vee * \vee *} = T$ for all $T \in \mathrm{Mor}(U,V)$.

(ii) The contravariant functor F defined by $F(U) = \bar{U}$ and $F(T) = T^\vee$ can be made into a tensor functor by defining $F_2(U,V) \in \mathrm{Mor}(\bar{V} \otimes \bar{U}, \overline{U \otimes V})$ by the identity

$$(F_2(U,V) \otimes \iota \otimes \iota)(\iota \otimes R_U \otimes \iota)R_V = R_{U\otimes V}.$$

Example 2.2.23. — Let G be a compact quantum group, and U a finite dimensional unitary representation. By Example 2.2.3 the operators $R = (1 \otimes \rho_{\bar{U}}^{-1/2})r$ and $\bar{R} = (\rho_U^{1/2} \otimes 1)\bar{r}$ solve the conjugate equations for U. Consider $(R, \bar{R})$ as a solution of the conjugate equations for H_U in the category Hilb_f and compute $T^\vee$ for $T \in B(H_U)$. We have $(1 \otimes T)r = (j(T) \otimes 1)r$. It follows that $T^\vee = j(\rho_U^{1/2} T \rho_U^{-1/2})$.

Consider now $T = \pi_U(\omega)$, where $\omega \in \mathscr{U}(G)$ and $\pi_U \colon \mathscr{U}(G) \to B(H_U)$ is the representation defined by U, so $\pi_U(\omega) = (\iota \otimes \omega)(U)$. By (1.7.1) we have $\pi_{\bar{U}}(\omega) = j(\pi_U(\hat{R}(\omega)))$, where $\hat{R}$ is the unitary antipode on $\mathscr{U}(G)$. It follows that

$$\pi_U(\omega)^\vee = j(\pi_U(\rho^{1/2}\omega\rho^{-1/2})) = \pi_{\bar{U}}(\hat{R}(\rho^{1/2}\omega\rho^{-1/2})) = \pi_{\bar{U}}(\hat{S}(\omega)). \qquad \square$$

References. — [**46**], [**59**].

2.3. FIBER FUNCTORS AND RECONSTRUCTION THEOREMS

Let $\mathscr{C}$ be a C*-tensor category.

Definition 2.3.1. — A tensor functor $F \colon \mathscr{C} \to \mathrm{Hilb}_f$ is called a fiber functor if it is faithful (that is, injective on morphisms) and exact.

We will mainly deal with C*-tensor categories with conjugates. Then every object is a direct sum of simple ones, hence every exact sequence splits, and therefore any linear functor $\mathscr{C} \to \mathrm{Hilb}_f$ is exact. Furthermore, in this case a linear functor is faithful if and only if the image of every simple object is nonzero. This is automatically true for tensor functors $F \colon \mathscr{C} \to \mathrm{Hilb}_f$, since for every nonzero object U the unit object is a subobject of $\bar{U} \otimes U$, and therefore $\mathbb{C}$ embeds into $F(\bar{U} \otimes U) \cong F(\bar{U}) \otimes F(U)$, so $F(U) \neq 0$. Therefore for C*-tensor categories with conjugates a fiber functor is simply a tensor functor $F \colon \mathscr{C} \to \mathrm{Hilb}_f$.

Let G be a compact quantum group. The simplest example of a unitary fiber functor on $\mathrm{Rep}\, G$ is the one defined by letting $F(U) = H_U$ for every finite dimensional unitary representation of G, while F_2, as well the action of F on morphisms, are taken to be the identity maps. We call it the canonical fiber functor on $\mathrm{Rep}\, G$.

***Theorem 2.3.2* (Woronowicz's Tannaka-Krein duality).** — *Let $\mathscr{C}$ be a C*-tensor category with conjugates, $F \colon \mathscr{C} \to \mathrm{Hilb}_f$ be a unitary fiber functor. Then there exist a compact quantum group G and a unitary monoidal equivalence $E \colon \mathscr{C} \to \mathrm{Rep}\, G$ such that F is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor* $\mathrm{Rep}\, G \to \mathrm{Hilb}_f$

with E. Furthermore, the Hopf $*$-algebra $(\mathbb{C}[G],\Delta)$ for such a G is uniquely determined up to isomorphism.

We may assume that $\mathscr{C}$ is strict. Replacing F by a naturally unitarily monoidally isomorphic functor we may also assume that $F(\mathbb{1})=\mathbb{C}$ and F_0 is the identity map.

Consider the $*$-algebra $\mathrm{End}(F) = \mathrm{Nat}(F,F)$ of natural transformations from F to F, where we consider F simply as a functor, ignoring its tensor structure. Explicitly this algebra can be described as follows. Choose representatives U_α of the isomorphism classes of simple objects. Since every object in $\mathscr{C}$ is a direct sum of U_α, a natural transformation $\eta\colon F\to F$ is completely determined by the maps $\eta_{U_\alpha}\colon F(U_\alpha)\to F(U_\alpha)$. Therefore

$$\mathrm{End}(F)\cong\prod_\alpha B(F(U_\alpha)).$$

Using the tensor structure on F we can define a $*$-homomorphism

$$\delta\colon \mathrm{End}(F)\to\mathrm{End}(F^{\otimes 2})\cong\prod_{\alpha,\beta}B(F(U_\alpha)\otimes F(U_\beta))$$

such that $\delta(\eta)$ is determined by the commutative diagrams

$$\begin{tikzcd} F(U)\otimes F(V) \arrow[r, "\delta(\eta)_{U,V}"] \arrow[d, "F_2"'] & F(U)\otimes F(V) \arrow[d, "F_2"] \\ F(U\otimes V) \arrow[r, "\eta_{U\otimes V}"'] & F(U\otimes V) \end{tikzcd},$$

so $\delta(\eta)_{U,V} = F_2^*\eta_{U\otimes V}F_2$. The homomorphisms $\iota\otimes\delta$ and $\delta\otimes\iota$ extend to homomorphisms $\mathrm{End}(F^{\otimes 2})\to\mathrm{End}(F^{\otimes 3})$. Then $(\iota\otimes\delta)\delta=(\delta\otimes\iota)\delta$ by definition of a tensor functor.

We will see that $(\mathrm{End}(F),\delta)\cong(\mathscr{U}(G),\hat\Delta)$ for some G. But first we need to define an analogue of the antipode $\hat S$. In view of Example 2.2.23 the formula is easy to guess. Let us first introduce the following notation. For $T\in\mathrm{Mor}(U,V\otimes W)$ denote by $\Theta(T)$ the map $F_2^*F(T)\colon F(U)\to F(V)\otimes F(W)$. If $(R,\bar R)$ solves the conjugate equations for U and $\bar U$, then $(\Theta(R),\Theta(\bar R))$ solves the conjugate equations for $F(U)$ and $F(\bar U)$.

Lemma 2.3.3. — *For every $\eta\in\mathrm{End}(F)$ there exists a unique element $\eta^\vee\in\mathrm{End}(F)$ such that if $(R,\bar R)$ solves the conjugate equations for U and $\bar U$, then $(\eta^\vee)_{\bar U}=(\eta_U)^\vee$, where $(\eta_U)^\vee$ is computed using the solution $(\Theta(R),\Theta(\bar R))$ of the conjugate equations for $F(U)$ and $F(\bar U)$.*

Proof. — First observe that for fixed $\bar U$ the map $(\eta_U)^\vee$ does not depend on the solution of the conjugate equations for U and $\bar U$. Indeed, by Proposition 2.2.5 any other solution $(R',\bar R')$ has the form $R'=(\iota\otimes T^*)R$, $\bar R'=(T^{-1}\otimes\iota)\bar R$. Then

$$\Theta(R')=(1\otimes F(T^*))\Theta(R)$$

Since η_U commutes with $F(T^*)$ by the naturality of η, we conclude that $(\eta_U)^\vee$ is independent of whether we use $\Theta(R)$ or $\Theta(R')$ to define it.

Next using the naturality of η it is easy to check that if for two isomorphic objects U and U' we choose the same conjugate object, then $(\eta_U)^\vee = (\eta_{U'})^\vee$. We therefore get a well-defined collection of maps $(\eta^\vee)_V : F(V) \to F(V)$ such that if V is conjugate to U, then $(\eta^\vee)_V = (\eta_U)^\vee$. It remains to check that these maps are natural in V.

Any morphism $S\colon V_1 \to V_2$ equals $T^\vee$ for some morphism $T\colon U_2 \to U_1$ and a fixed choice of solutions $(R_1, \bar{R}_1)$ and $(R_2, \bar{R}_2)$ of the conjugate equations for (U_1, V_1) and (U_2, V_2). From $\eta_{U_1} F(T) = F(T)\eta_{U_2}$ we get

$$F(T)^\vee (\eta_{U_1})^\vee = (\eta_{U_2})^\vee F(T)^\vee,$$

where we use $\Theta(R_1)$ and $\Theta(R_2)$ to define $F(T)^\vee$. Using the easily verifiable identities $F(T)^\vee = F(T^\vee) = F(S)$, we therefore get $F(S)(\eta^\vee)_{V_1} = (\eta^\vee)_{V_2} F(S)$. $\square$

We are now ready to define a candidate $(\mathscr{A}, \Delta)$ for $(\mathbb{C}[G], \Delta)$. We take $\mathscr{A}$ to be the subspace of the dual space $\mathrm{End}(F)^*$ consisting of the elements a such that $a(\eta)$ only depends on the operators η_U for finitely many objects U. More concretely, if we identify $\mathrm{End}(F)$ with $\prod_\alpha B(F(U_\alpha))$, then $\mathscr{A} = \oplus_\alpha B(F(U_\alpha))^*$. If $a, b \in \mathscr{A}$, then $a \otimes b$ is a well-defined element of $\mathrm{End}(F^{\otimes 2})^*$. Therefore we can define a product on $\mathscr{A}$ by

$$ab = (a \otimes b)\delta.$$

Note that if $a \in B(F(U_\alpha))^*$ and $b \in B(F(U_\beta))^*$, then $ab \in \oplus_\gamma B(F(U_\gamma))^*$, where the sum is taken of the finite set of indices γ such that $\mathrm{Mor}(U_\gamma, U_\alpha \otimes U_\beta) \neq 0$. Since δ is coassociative, the product on $\mathscr{A}$ is associative. The algebra $\mathscr{A}$ is unital, with unit given by $1(\eta) = \eta_{\mathbb{1}} \in \mathrm{End}(\mathbb{C}) = \mathbb{C}$. Define a comultiplication $\Delta\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ by

$$\Delta(a)(\omega \otimes \eta) = a(\omega\eta).$$

This is a unital coassociative homomorphism. Define also a character $\varepsilon\colon \mathscr{A} \to \mathbb{C}$ and a linear map $S\colon \mathscr{A} \to \mathscr{A}$ by

$$\varepsilon(a) = a(1) \ \text{ and } \ S(a)(\eta) = a(\eta^\vee).$$

Lemma 2.3.4. — *$(\mathscr{A}, \Delta)$ is a Hopf algebra with counit ε and antipode S.*

In proving this lemma it will be convenient to use the following convention, which will also be useful later. We will work with elements $\delta(\eta)$ as if they were finite sums of elementary tensors. All the computations will be valid, since for any finite set of indices α_i the image of $\delta(\eta)$ in $\oplus_{i,j} B(F(U_{\alpha_i}) \otimes F(U_{\alpha_j}))$ coincides with the image of a finite sum of elementary tensors. Furthermore, we will omit sums and simply write $\delta(\eta) = \eta_{(1)} \otimes \eta_{(2)}$. This is called Sweedler's sumless notation.

Proof of Lemma 2.3.4. — The identities $(\iota\otimes\varepsilon)\Delta(a) = a$ and $(\varepsilon\otimes\iota)\Delta(a) = a$ are verified immediately by applying $\eta \in \mathrm{End}(F)$ to both sides. The identity $m(\iota\otimes S)\Delta(a) = \varepsilon(a)1$ for all a is in turn equivalent to

$$\eta_{(1)}\eta^{\vee}_{(2)} = \eta_{\mathbb{1}}1 \text{ for all } \eta \in \mathrm{End}(F). \tag{2.3.1}$$

Fix an object U and a solution $(R, \bar{R})$ of the conjugate equations for U. Using the definition of ${}^{\vee}$, $(1\otimes T)\Theta(R) = (T^{\vee}\otimes 1)\Theta(R)$, we compute:

$$((\eta_{(1)})_{\bar{U}}(\eta^{\vee}_{(2)})_{\bar{U}}\otimes\iota)\Theta(R) = \delta(\eta)_{\bar{U},U}\Theta(R) = F_2^*\eta_{\bar{U}\otimes U}F(R) = \Theta(R)\eta_{\mathbb{1}}.$$

Hence $(\eta_{(1)}\eta^{\vee}_{(2)})_{\bar{U}} = \eta_{\mathbb{1}}1 \in B(F(\bar{U}))$. Since this is true for all U, we conclude that (2.3.1) holds. Similarly, using the identity $\Theta(\bar{R})^*(T\otimes 1) = \Theta(\bar{R})^*(1\otimes T^{\vee})$ for ${}^{\vee}$, one checks that $m(S\otimes\iota)\Delta(a) = \varepsilon(a)1$. ☐

Next define an antilinear map $a\mapsto a^*$ by

$$a^*(\eta) = \bar{a}(\eta^{\vee}) = \overline{a(\eta^{\vee *})}.$$

Since $\eta^{\vee*\vee*} = \eta$ by Remark 2.2.22(i), we have $a^{**} = a$. It is also straightforward to check that the map Δ is $*$-preserving. On other hand, in order to show that $*$ is anti-multiplicative we need to know that $\delta(\eta^{\vee}) = ({}^{\vee}\otimes{}^{\vee})\delta^{\mathrm{op}}(\eta)$. This is not difficult to check directly. Alternatively, on the dual side this is equivalent to anti-multiplicativity of the antipode S on $\mathscr{A}$, and this holds in any Hopf algebra. Therefore $(\mathscr{A}, \Delta)$ is a Hopf $*$-algebra.

For every U in $\mathscr{C}$ define an element $X^U \in B(F(U))\otimes\mathrm{End}(F)^*$ by requiring

$$(\iota\otimes\eta)(X^U) = \eta_U \text{ for all } \eta \in \mathrm{End}(F).$$

Clearly, $X^U \in B(F(U))\otimes\mathscr{A}$.

Lemma 2.3.5. — *We have:*

(i) *the elements X^U are unitary corepresentations of $(\mathscr{A}, \Delta)$;*

(ii) *if $T \in \mathrm{Mor}(U, V)$, then $(F(T)\otimes 1)X^U = X^V(F(T)\otimes 1)$;*

(iii) $(F_2\otimes 1)X^U_{13}X^V_{23} = X^{U\otimes V}(F_2\otimes 1)$.

Proof. — (i) The identity $(\iota\otimes\varepsilon)(X^U) = 1$ is immediate by definition, since $\varepsilon = 1 \in \mathrm{End}(F)$. The identity $(\iota\otimes\Delta)(X^U) = X^U_{12}X^U_{13}$ is checked by applying $\iota\otimes\omega\otimes\eta$ to both sides.

Therefore X^U is a corepresentation of $(\mathscr{A}, \Delta)$. As we already observed in Section 1.6, this implies that X^U is invertible and $(\iota\otimes S)(X^U) = (X^U)^{-1}$. Hence, for every $\eta \in \mathrm{End}(F)$,

$$\begin{aligned}(\iota\otimes\eta)((X^U)^{-1}) &= (\iota\otimes\eta^{\vee})(X^U) = (\eta^{\vee})_U = (\eta^{\vee*})^*_U\\ &= (\iota\otimes\eta^{\vee*})(X^U)^* = (\iota\otimes\eta)((X^U)^*).\end{aligned}$$

It follows that $(X^U)^{-1} = (X^U)^*$.

(ii) This becomes obvious upon applying $\iota \otimes \eta$ to both sides.

(iii) Applying $\iota \otimes \iota \otimes \eta$ to the left hand side we obtain

$$F_2(\iota \otimes \iota \otimes \eta)(X^U_{13}X^V_{23}) = F_2(\iota \otimes \iota \otimes \hat\delta(\eta))(X^U_{13}X^V_{24}) = F_2\hat\delta(\eta)_{U,V} = \eta_{U\otimes V}F_2,$$

which is exactly what we get if we apply $\iota \otimes \iota \otimes \eta$ to $X^{U\otimes V}(F_2 \otimes 1)$. □

It is now easy to complete the proof of Woronowicz's theorem.

Proof of Theorem 2.3.2. — By Theorem 1.6.7, $(\mathscr{A}, \Delta) = (\mathbb{C}[G], \Delta)$ for a compact quantum group G. Identities (ii) and (iii) in Lemma 2.3.5 show then that we can define a unitary tensor functor $E\colon \mathscr{C} \to \operatorname{Rep} G$ by letting $E(U) = X^U$, $E(T) = F(T)$ on morphisms, and $E_2 = F_2$. Then F is the composition of the canonical fiber functor $\operatorname{Rep} G \to \mathrm{Hilb}_f$ with E. To show that E is a unitary monoidal equivalence it suffices to check that the representations X^{U_α} of G are irreducible, pairwise nonequivalent, and that they exhaust the equivalence classes of irreducible representations of G. But all these properties follow immediately from the fact that matrix coefficients of the representations X^{U_α} form a basis in $\oplus_\alpha B(F(U_\alpha))^* = \mathscr{A}$.

The uniqueness follows by observing that if G' is a compact quantum group and

$$F'\colon \operatorname{Rep} G' \to \mathrm{Hilb}_f$$

is the canonical fiber functor, then $(\mathscr{U}(G'), \hat\Delta)$ can be identified with $(\operatorname{End}(F'), \hat\delta)$ constructed as above: an element $\omega \in \mathscr{U}(G')$ considered as an element of $\operatorname{End}(F')$ acts on $F'(U) = H_U$ by $\pi_U(\omega)$. In order to see that this is indeed enough, note that if $E'\colon \mathscr{C} \to \operatorname{Rep} G'$ is a unitary monoidal equivalence such that $F'E'$ is naturally unitarily monoidally isomorphic to F, then using this isomorphism of functors we get an isomorphism

$$(\mathscr{U}(G), \hat\Delta) = (\operatorname{End}(F), \hat\delta) \cong (\operatorname{End}(F'), \hat\delta) = (\mathscr{U}(G'), \hat\Delta),$$

and from this obtain $(\mathbb{C}[G], \Delta) \cong (\mathbb{C}[G'], \Delta)$ by duality. We leave the details to the reader. □

Remark 2.3.6. — Theorem 1.6.7 is more than what is really needed for the proof of Theorem 2.3.2. Namely, the key point of Theorem 1.6.7 is that under certain assumptions a Hopf $*$-algebra admits a faithful state. In the present case we could define such a state by letting $h(1) = 1$ and $(\iota \otimes h)(X^{U_\alpha}) = 0$ if $U_\alpha \not\cong \mathbb{1}$, and then check positivity and faithfulness using properties of conjugate objects. We will do this in a more general setting in the proof of Theorem 2.3.11.

Note that not every C*-tensor category admits a fiber functor.

Example 2.3.7. — Consider the category $\mathscr{C}(\Gamma,\omega)$ from Example 2.1.2. Then by Example 2.2.4 the object $V_{g^{-1}}$ is conjugate to V_g, so $\mathscr{C}(\Gamma,\omega)$ is a C*-tensor category with conjugates. Assume $F\colon \mathscr{C}(\Gamma,\omega)\to \mathrm{Hilb}_f$ is a fiber functor. Since $V_g\otimes V_{g^{-1}}=V_e=\mathbb{1}$, the spaces $F(V_g)$ are one-dimensional. For every $g\in\Gamma$ fix a nonzero vector $\xi_g\in F(V_g)$. Then $F_2\colon F(V_g)\otimes F(V_h)\to F(V_{gh})$ maps $\xi_g\otimes\xi_h$ into $c(g,h)\xi_{gh}$, and $\omega=\partial c$. Therefore a fiber functor on $\mathscr{C}(\Gamma,\omega)$ exists if and only if ω is a coboundary. □

A C*-tensor category can also have fiber functors producing nonisomorphic quantum groups.

Definition 2.3.8. — Compact quantum groups G_1 and G_2 are called monoidally equivalent if the categories $\mathrm{Rep}\, G_1$ and $\mathrm{Rep}\, G_2$ are unitarily monoidally equivalent.

Example 2.3.9. — Assume G is a compact quantum group. By a $\mathbb{T}$-valued 2-cocycle on $\hat{G}$ we mean a unitary element $\mathscr{E}\in\mathscr{U}(G\times G)$ such that

$$(\mathscr{E}\otimes 1)(\hat{\Delta}\otimes\iota)(\mathscr{E})=(1\otimes\mathscr{E})(\iota\otimes\hat{\Delta})(\mathscr{E}). \tag{2.3.2}$$

Given such a cocycle, we define a unitary fiber functor $F\colon\ \mathrm{Rep}\, G\to\mathrm{Hilb}_f$ such that it is the identity on objects and morphism, while the tensor structure F_2 is given by the action of $\mathscr{E}^*$. We will discuss this in more detail in the next chapter. The functor F defines a new compact quantum group $G_{\mathscr{E}}$. More explicitly, the $*$-algebra $\mathscr{U}(G_{\mathscr{E}})$ can be identified with $\mathscr{U}(G)$, but the coproduct is different, given by

$$\hat{\Delta}_{\mathscr{E}}(\omega)=\mathscr{E}\hat{\Delta}(\omega)\mathscr{E}^*.$$

It follows that we can identify $\mathbb{C}[G_{\mathscr{E}}]$ with $\mathbb{C}[G]$ as vector spaces, and the coproduct Δ remains the same, while the new product is given by

$$a\cdot_{\mathscr{E}} b=m(\mathscr{E}^* * (a\otimes b) * \mathscr{E}),$$

where $\omega * a=(\iota\otimes\omega)\Delta(a)$ and $a*\omega=(\omega\otimes\iota)\Delta(a)$. The $*$-structure on $\mathbb{C}[G_{\mathscr{E}}]$ is more difficult to describe, we will return to this in a moment.

By construction the quantum groups $G_{\mathscr{E}}$ and G are monoidally equivalent. Now, if G is a genuine group, the new coproduct $\hat{\Delta}_{\mathscr{E}}$ is in general non-cocommutative, so $G_{\mathscr{E}}$ is not a group. Therefore a group can be monoidally equivalent to a genuine quantum group. It is more difficult, but possible, to give examples where $G_{\mathscr{E}}$ is again a group, but it is nonisomorphic to G [**32, 43**]. To some extent it is possible to describe completely all pairs of monoidally equivalent compact groups [**32, 43, 69**].

The most nontrivial part of the proof of Theorem 2.3.2 is the construction of the antipode using conjugate objects. It is therefore interesting to express the new antipode $\hat{S}_{\mathscr{E}}$ on $\mathscr{U}(G_{\mathscr{E}})=\mathscr{U}(G)$ explicitly in terms of $\mathscr{E}$. Since we will consider a similar problem in Section 4.4, we will omit the computations and only give the final answer. Consider the element $u=m(\iota\otimes\hat{S})(\mathscr{E})$. Then u is invertible, with inverse $u^{-1}=m(\hat{S}\otimes$

$\iota)(\mathscr{E}^*) = \hat{S}(u^*)$, and we have $\hat{S}_{\mathscr{E}} = u\hat{S}(\cdot)u^{-1}$. From this we can deduce a formula for the $*$-structure on $\mathbb{C}[G_{\mathscr{E}}]$. Namely, if we denote the involution on $\mathbb{C}[G_{\mathscr{E}}]$ by $*_{\mathscr{E}}$, then, using that $a^{*_{\mathscr{E}}}(\omega) = \overline{a(\hat{S}_{\mathscr{E}}(\omega)^*)}$, we get

$$a^{*_{\mathscr{E}}} = (u^* * a * (u^{-1})^*)^*.$$

Finally, we remark that the above formulas of course make sense for any Hopf algebra, independently of whether it comes from a compact quantum group or not. Thus, given a Hopf algebra $(\mathscr{U}, \hat{\Delta})$ and an invertible element $\mathscr{E} \in \mathscr{U} \otimes \mathscr{U}$ satisfying (2.3.2), we can define a new Hopf algebra $(\mathscr{U}_{\mathscr{E}}, \hat{\Delta}_{\mathscr{E}})$, called a twisting of $(\mathscr{U}, \hat{\Delta})$, with $\mathscr{U}_{\mathscr{E}} = \mathscr{U}$ as algebras and with $\hat{\Delta}_{\mathscr{E}} = \mathscr{E}\hat{\Delta}(\cdot)\mathscr{E}^{-1}$. Its antipode is given by $\hat{S}_{\mathscr{E}} = u\hat{S}(\cdot)u^{-1}$, where $u = m(\iota \otimes \hat{S})(\mathscr{E})$ is invertible, with inverse given by $u^{-1} = m(\hat{S} \otimes \iota)(\mathscr{E}^{-1})$. □

Example 2.3.10. — In Section 2.5 we will show that two free orthogonal compact quantum groups $A_o(F)$ and $A_o(F')$, defined in Example 1.1.7, are monoidally equivalent if and only if $F\bar{F}$ and $F'\bar{F}'$ have the same sign and $\mathrm{Tr}(F^*F) = \mathrm{Tr}(F'^*F')$, while they are isomorphic if and only if F and F' have the same size and $F' = vFv^t$ for a unitary matrix v. □

A key tool to understand the relation between monoidally equivalent quantum groups is a linking algebra, also known as a Hopf-bi-Galois object in the algebraic literature, which we will now introduce.

Assume that $\mathscr{C}$ is a strict C*-tensor category with conjugates, and $E, F\colon \mathscr{C} \to \mathrm{Hilb}_f$ are two unitary fiber functors. As above, assume $E(\mathbb{1}) = F(\mathbb{1}) = \mathbb{C}$ and $E_0 = F_0 = \iota$. For $T \in \mathrm{Mor}(U, V \otimes W)$ put $\Theta_E(T) = E_2^* E(T) \in B(E(U), E(V) \otimes E(W))$ and $\Theta_F(T) = F_2^* F(T)$.

Consider the space $\mathrm{Nat}(E, F)$ of natural transformations from E to F. It can be identified with $\prod_\alpha B(E(U_\alpha), F(U_\alpha))$. We can then define a linear map

$$\delta\colon\ \mathrm{Nat}(E, F) \to \mathrm{Nat}(E^{\otimes 2}, F^{\otimes 2})$$

by $\delta(\eta)_{U,V} = F_2^* \eta_{U\otimes V} E_2$. Let $\mathscr{B} \subset \mathrm{Nat}(E, F)^*$ be the subspace consisting of elements b such that $b(\eta)$ depends only on the operators η_U for finitely many objects U. In other words, $\mathscr{B} = \oplus_\alpha B(E(U_\alpha), F(U_\alpha))^*$.

Define a product and an involution on $\mathscr{B}$ by

$$(bc)(\eta) = (b \otimes c)\delta(\eta), \quad b^*(\eta) = \overline{b(\eta^{\vee *})}.$$

Note that $^\vee$ is now a well-defined map $\mathrm{Nat}(E, F) \to \mathrm{Nat}(F, E)$. For every object U define also an element $X^U \in B(E(U), F(U)) \otimes \mathscr{B}$ by

$$(\iota \otimes \eta)(X^U) = \eta_U.$$

Theorem 2.3.11. — *With the above notation we have:*

(i) *$\mathscr{B}$ is a unital $*$-algebra spanned by the matrix coefficients of the elements X^U; these elements are unitary and they satisfy the following properties:*

(a) *if $T \in \mathrm{Mor}(U,V)$, then $(F(T)\otimes 1)X^U = X^V(E(T)\otimes 1)$;*

(b) *$(F_2\otimes 1)X^U_{13}X^V_{23} = X^{U\otimes V}(E_2\otimes 1)$;*

(c) *if $(R,\bar R)$ is a solution of the conjugate equations for U and $\bar U$, then*

$$(X^U_{13})^*(\Theta_F(\bar R)\otimes 1) = X^{\bar U}_{23}(\Theta_E(\bar R)\otimes 1),$$
$$(\Theta_E(R)^*\otimes 1)(X^U_{23})^* = (\Theta_F(R)^*\otimes 1)X^{\bar U}_{13};$$

(ii) *the linear functional φ on $\mathscr{B}$ defined by $\varphi(1) = 1$ and $(\iota\otimes\varphi)(X^{U_\alpha}) = 0$ for $U_\alpha \not\cong \mathbb{1}$ satisfies the following orthogonality relations: if $(R_\alpha,\bar R_\alpha)$ is a solution of the conjugate equations for U_α, then*

$$(\iota\otimes\varphi)((X^{U_\alpha})^*(T\otimes 1)X^{U_\beta}) = \delta_{\alpha\beta}\|R_\alpha\|^{-2}\Theta_F(R_\alpha)^*(1\otimes T)\Theta_F(R_\alpha)1$$

for all $T\in B(F(U_\beta),F(U_\alpha))$, and

$$(\iota\otimes\varphi)(X^{U_\alpha}(T\otimes 1)(X^{U_\beta})^*) = \delta_{\alpha\beta}\|\bar R_\alpha\|^{-2}\Theta_E(\bar R_\alpha)^*(T\otimes 1)\Theta_E(\bar R_\alpha)1$$

for all $T\in B(E(U_\beta),E(U_\alpha))$; in particular, φ is a faithful state on $\mathscr{B}$ and therefore $\mathscr{B}$ admits a C^*-completion.

Proof. — (i) Parts (a) and (b), as well as the statement that $\mathscr{B}$ is a unital algebra spanned by the matrix coefficients of the elements X^U, are simple and can be proved in the same way as the similar statements in the proof of Theorem 2.3.2. Therefore it remains to check that the involution is anti-multiplicative, the elements X^U are unitary and the identities in (c) hold. We will start by proving (c).

For any $\eta\in\mathrm{Nat}(E,F)$ we have

$$(\iota\otimes\eta)((X^U)^*) = (\iota\otimes\eta^{\vee *})(X^U)^* = (\eta^{\vee *})^*_U = (\eta^\vee)_U.$$

Hence, if $(R,\bar R)$ is a solution of the conjugate equations for U, we can rewrite the identity $((\eta^\vee)_U\otimes 1)\Theta_F(\bar R) = (1\otimes\eta_{\bar U})\Theta_E(\bar R)$ for all η as

$$(X^U_{13})^*(\Theta_F(\bar R)\otimes 1) = X^{\bar U}_{23}(\Theta_E(\bar R)\otimes 1).$$

The second identity in (c) can be proved similarly, but it also follows from the first one by taking adjoints and replacing U by $\bar U$.

Next, using properties (a)-(c), we compute:

$$\begin{aligned}X^U_{13}(X^U_{13})^*(\Theta_F(\bar R)\otimes 1) &= X^U_{13}X^{\bar U}_{23}(\Theta_E(\bar R)\otimes 1) = (F_2^*\otimes 1)X^{U\otimes\bar U}(E(\bar R)\otimes 1)\\ &= (F_2^*\otimes 1)(F(\bar R)\otimes 1)X^{\mathbb{1}} = \Theta_F(\bar R)\otimes 1.\end{aligned}$$

Hence $X^U(X^U)^* = 1$. The identity $(X^U)^*X^U = 1$ is proved similarly.

It remains to check that the involution is anti-multiplicative. For any objects U and V we compute:

$$\begin{aligned}(X^U_{13}X^V_{23})^*(F_2^*\otimes 1) &= (E_2^*\otimes 1)(X^{U\otimes V})^* = (E_2^*\otimes 1)(X^{U\otimes V})^{-1}\\ &= (X^V_{23})^{-1}(X^U_{13})^{-1}(F_2^*\otimes 1) = (X^V_{23})^*(X^U_{13})^*(F_2^*\otimes 1).\end{aligned}$$

Therefore $(X^U_{13}X^V_{23})^* = (X^V_{23})^*(X^U_{13})^*$. This implies that the involution is anti-multiplicative, since the matrix coefficients of X^{U_α} span $\mathscr{B}$.

(ii) Given indices α and β, a decomposition of $\bar U_\alpha\otimes U_\beta$ into simple objects is defined by isometries $w_i\colon U_{\alpha_i}\to \bar U_\alpha\otimes U_\beta$ such that $\sum_i w_iw_i^*=1$. Then

$$X^{\bar U_\alpha\otimes U_\beta} = \sum_i (F(w_i)\otimes 1)X^{U_{\alpha_i}}(E(w_i)^*\otimes 1).$$

It follows that if $\alpha=\beta$, so that the only copy of $\mathbb{1}$ in $\bar U_\alpha\otimes U_\alpha$ is given by the isometry $\|R_\alpha\|^{-1}R_\alpha$, then $(\iota\otimes\varphi)(X^{\bar U_\alpha\otimes U_\alpha}) = \|R_\alpha\|^{-2}F(R_\alpha)E(R_\alpha)^*$. By (i)(b) we can write this as

$$(\iota\otimes\iota\otimes\varphi)(X^{\bar U_\alpha}_{13}X^{U_\alpha}_{23}) = \|R_\alpha\|^{-2}\Theta_F(R_\alpha)\Theta_E(R_\alpha)^*.$$

Using (i)(c), for any $T\in B(F(U_\alpha))$ we then get

$$\begin{aligned}&\Theta_E(R_\alpha)^*\big(1\otimes(\iota\otimes\varphi)((X^{U_\alpha})^*(T\otimes 1)X^{U_\alpha})\big)\\ &\qquad= (\iota\otimes\iota\otimes\varphi)\big((\Theta_E(R_\alpha)^*\otimes 1)(X^{U_\alpha})^*_{23}(1\otimes T\otimes 1)X^{U_\alpha}_{23}\big)\\ &\qquad= (\iota\otimes\iota\otimes\varphi)\big((\Theta_F(R_\alpha)^*\otimes 1)X^{\bar U_\alpha}_{13}(1\otimes T\otimes 1)X^{U_\alpha}_{23}\big)\\ &\qquad= \|R_\alpha\|^{-2}\Theta_F(R_\alpha)^*(1\otimes T)\Theta_F(R_\alpha)\Theta_E(R_\alpha)^*.\end{aligned}$$

This gives the first orthogonality relation in (ii) for $\alpha=\beta$. On the other hand, if $\alpha\ne\beta$, then $(\iota\otimes\varphi)(X^{\bar U_\alpha\otimes U_\beta})=0$, and a computation similar to the one above shows that $(\iota\otimes\varphi)((X^{U_\alpha})^*(T\otimes 1)X^{U_\beta})=0$ for any $T\in B(F(U_\beta),F(U_\alpha))$.

The second orthogonality relation is proved similarly.

As in the proof of Theorem 1.6.7, either of the two orthogonality relations implies that $\varphi(b^*b)>0$ for all nonzero $b\in\mathscr{B}$. □

Consider now the compact quantum groups G_E and G_F defined by the functors E and F. The space $\mathrm{Nat}(E,F)$ is an $\mathrm{End}(F)$-$\mathrm{End}(E)$-bimodule. By taking the duals of the maps $\mathrm{End}(F)\otimes\mathrm{Nat}(E,F)\to\mathrm{Nat}(E,F)$ and $\mathrm{Nat}(E,F)\otimes\mathrm{End}(E)\to\mathrm{Nat}(E,F)$ we get unital $*$-homomorphisms

$$\alpha_F\colon\mathscr{B}\to\mathbb{C}[G_F]\otimes\mathscr{B}\ \text{ and }\ \alpha_E\colon\mathscr{B}\to\mathscr{B}\otimes\mathbb{C}[G_E]$$

such that $(\iota\otimes\alpha_F)\alpha_F=(\Delta_F\otimes\iota)\alpha_F$ and $(\alpha_E\otimes\iota)\alpha_E=(\iota\otimes\Delta_E)\alpha_E$, where Δ_E and Δ_F denote the comultiplications on $\mathbb{C}[G_E]$ and $\mathbb{C}[G_F]$. Explicitly, let $X^U_E\in B(E(U))\otimes\mathbb{C}[G_E]$ and $X^U_F\in B(F(U))\otimes\mathbb{C}[G_F]$ be the unitaries constructed in the proof of Theorem 2.3.2. Then

$$(\iota\otimes\alpha_F)(X^U) = (X^U_F)_{12}X^U_{13}\ \text{ and }\ (\iota\otimes\alpha_E)(X^U) = X^U_{12}(X^U_E)_{13}.$$

The linking algebra $\mathscr{B}$ equipped with the actions α_E and α_F of G_E and G_F allows one to deduce a lot of properties of one quantum group from the other. We refer the reader to [**23, 81, 88**] for examples.

Our main examples of C*-tensor categories will appear as categories of modules. Let $(\mathscr{U},\hat\Delta)$ be a Hopf $*$-algebra. Denote the counit by $\hat\varepsilon$ and the antipode by $\hat S$. Consider the category $\operatorname{Rep}\mathscr{U}$ of finite dimensional unital $*$-representations of $\mathscr{U}$. It is a C*-tensor category, with the tensor product of representations $\pi\colon \mathscr{U}\to B(H_\pi)$ and $\vartheta\colon\mathscr{U}\to B(H_\vartheta)$ defined by

$$(\pi\otimes\vartheta)(\omega)=(\pi\otimes\vartheta)\hat\Delta(\omega).$$

The counit $\hat\varepsilon$ is the unit object in $\operatorname{Rep}\mathscr{U}$.

For a finite dimensional representation $\pi\colon\mathscr{U}\to B(H)$ define the contragredient representation $\pi^c\colon\mathscr{U}\to B(H^*)$ by

$$(\pi^c(\omega)f)(v)=f(\pi(\hat S(\omega))v)\ \text{ for }\ \omega\in\mathscr{U}^*,\ f\in H^*,\ v\in H.$$

If H is a Hilbert space, then identifying H^* with $\bar H$ we have $\pi^c(\omega)=j(\pi(\hat S(\omega)))$, so $\pi^c(\omega)\bar\xi=\overline{\pi(\hat S(\omega))^*\xi}$. In general π^c is neither a $*$-representation, nor is it equivalent to a $*$-representation, and correspondingly not every object in $\operatorname{Rep}\mathscr{U}$ has a conjugate.

Lemma 2.3.12. — *For any finite dimensional unital $*$-representation $\pi\colon\mathscr{U}\to B(H)$ the following conditions are equivalent:*

(i) π *has a conjugate in* $\operatorname{Rep}\mathscr{U}$*;*

(ii) *the contragredient representation π^c is unitarizable, that is, it is equivalent to a $*$-representation;*

(iii) $\operatorname{Mor}(\pi,\pi^{cc})$ *contains a positive invertible operator.*

Proof. — (i)$\Rightarrow$(ii) Assume $\vartheta\colon\mathscr{U}\to B(H_\vartheta)$ is conjugate to π, with $(R,\bar R)$ solving the conjugate equations. Consider also the maps $r\colon\mathbb{C}\to\bar H\otimes H$ and $\bar r\colon\mathbb{C}\to H\otimes\bar H$ from Example 2.2.2. By Proposition 2.2.5 the map $T=(R^*\otimes1)(1\otimes\bar r)\colon H_\vartheta\to\bar H$ is a linear isomorphism. But it is easy to check that the map $\bar r$ is a morphism $\mathbb{1}\to\pi\otimes\pi^c$ (while r is a morphism $\mathbb{1}\to\pi^c\otimes\pi^{cc}$). It follows that $T\in\operatorname{Mor}(\vartheta,\pi^c)$, so π^c is unitarizable.

(ii)$\Rightarrow$(iii) By assumption there exists an invertible operator $T\in B(\bar H)$ such that the representation $\vartheta=T\pi^c(\cdot)T^{-1}$ is $*$-preserving. Then $T\in\operatorname{Mor}(\pi^c,\vartheta)$, hence $j(T)\in\operatorname{Mor}(\vartheta^c,\pi^{cc})$. One the other hand, starting with the identity

$$Tj(\pi(\hat S(\omega)))=\vartheta(\omega)T,$$

and applying j and taking the adjoints, we get

$$j(T)^*\pi(\hat S(\omega)^*)=j(\vartheta(\omega^*))j(T)^*.$$

Replacing ω by $\hat{S}(\omega)^*$ we see that $j(T)^* \in \mathrm{Mor}(\pi, \vartheta^c)$. Therefore $\mathrm{Mor}(\pi, \pi^{cc})$ contains the positive invertible operator $j(T)j(T)^*$.

(iii)$\Rightarrow$(i) By assumption there exists a positive invertible operator $\rho \in B(H)$ such that $\pi(\hat{S}^2(\omega)) = \rho\pi(\omega)\rho^{-1}$. It is then easy to check that the representation $\vartheta = j(\rho)^{1/2}\pi^c(\cdot)j(\rho)^{-1/2}$ is $*$-preserving. As we already remarked above, we have $r \in \mathrm{Mor}(\mathbb{1}, \pi^c \otimes \pi^{cc})$ and $\bar{r} \in \mathrm{Mor}(\mathbb{1}, \pi \otimes \pi^c)$. Therefore, as $j(\rho)^{1/2} \in \mathrm{Mor}(\pi^c, \vartheta)$ and $\rho \in \mathrm{Mor}(\pi, \pi^{cc})$, letting

$$R = (j(\rho)^{1/2} \otimes \rho^{-1})r = (1 \otimes \rho^{-1/2})r \in \mathrm{Mor}(\mathbb{1}, \vartheta \otimes \pi),$$
$$\bar{R} = (1 \otimes j(\rho)^{1/2})\bar{r} = (\rho^{1/2} \otimes 1)\bar{r} \in \mathrm{Mor}(\mathbb{1}, \pi \otimes \vartheta),$$

we see that ϑ is conjugate to π. $\square$

Taking a C*-tensor subcategory $\mathscr{C}$ of $\mathrm{Rep}\,\mathscr{U}$ with conjugates together with the canonical fiber functor on it, we get a compact quantum group. We will assume that $\mathscr{C}$ is full, that is, the space of morphisms is the same as in $\mathrm{Rep}\,\mathscr{U}$ for any pair of objects in $\mathscr{C}$. Note that by our standing assumptions on tensor categories this implies that the class of representations in $\mathscr{C}$ is closed, up to isomorphism, under taking direct sums and subrepresentations. Note also that by Lemma 2.3.12 the property of having conjugates is equivalent to requiring that with every representation π the category $\mathscr{C}$ contains a representation equivalent to π^c. In this case we do not need the full strength of Woronowicz's theorem, the corresponding quantum group is described via a duality result as follows.

Theorem 2.3.13. — *Assume $(\mathscr{U}, \hat{\Delta})$ is a Hopf $*$-algebra, and $\mathscr{C} \subset \mathrm{Rep}\,\mathscr{U}$ is a full C*-tensor subcategory with conjugates and $\mathbb{1} = \hat{\varepsilon}$. Let $\mathscr{A}$ be the subspace of $\mathscr{U}^*$ spanned by the matrix coefficients of all representations π in $\mathscr{C}$. Then $\mathscr{A}$ is a Hopf $*$-algebra with multiplication, involution and comultiplication defined by*

$$(ab)(\omega) = (a \otimes b)\hat{\Delta}(\omega), \quad a^*(\omega) = \overline{a(\hat{S}(\omega)^*)}, \quad \Delta(a)(\omega \otimes \nu) = a(\omega\nu).$$

Furthermore, $(\mathscr{A}, \Delta) = (\mathbb{C}[G], \Delta)$ for a compact quantum group G, and if for every π in $\mathscr{C}$ we define an element $U_\pi \in B(H_\pi) \otimes \mathscr{A}$ by $(\iota \otimes \omega)(U_\pi) = \pi(\omega)$ for $\omega \in \mathscr{U}$, then the functor $F\colon \mathscr{C} \to \mathrm{Rep}\,G$ defined by $F(\pi) = U_\pi$, $F(T) = T$ on morphisms and with $F_2 = \iota$, is a unitary monoidal equivalence of categories.

Proof. — The multiplication is well-defined on $\mathscr{A}$, since the product of matrix coefficients of finite dimensional representations of $\mathscr{U}$ is a matrix coefficient of the tensor product of the representations. To show that the involution is well-defined note that if $a \in \mathscr{A}$ is a matrix coefficient of a representation π in $\mathscr{C}$ then a^* is a matrix coefficient of π^c. By the assumption that $\mathscr{C}$ has conjugates, the representation π^c is equivalent to a representation in $\mathscr{C}$. Hence $a^* \in \mathscr{A}$. It is now easy to check that $\mathscr{A}$ is a unital $*$-algebra with unit $\hat{\varepsilon}$.

For every π in $\mathscr{C}$ we have $(\iota\otimes\Delta)(U_\pi) = (U_\pi)_{12}(U_\pi)_{13}$, since by applying $\iota\otimes\omega\otimes\nu$ we see that this identity is equivalent to multiplicativity of π. Therefore $\Delta\colon \mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$ is well-defined. Define also $S\colon \mathscr{A} \to \mathscr{A}$ by $S(a) = a\hat{S}$ and $\varepsilon\colon \mathscr{A} \to \mathbb{C}$ by $\varepsilon(a) = a(1)$. Note that S is well-defined for the same reason as that the involution is. It is now routine to check that $(\mathscr{A},\Delta)$ is a Hopf $*$-algebra with counit ε and antipode S; this is essentially what we claimed already in Section 1.6 when discussing dual Hopf $*$-algebras.

For every π in $\mathscr{C}$ we have $(\iota\otimes\varepsilon)(U_\pi) = \pi(1) = 1$. Therefore U_π is a corepresentation of $\mathscr{A}$. We now proceed as in the proof of Lemma 2.3.5(i). The element U_π is invertible and $(\iota\otimes S)(U_\pi) = U_\pi^{-1}$. Hence, for every $\omega \in \mathscr{U}$,

$$\begin{aligned}(\iota\otimes\omega)(U_\pi^{-1}) &= (\iota\otimes\hat{S}(\omega))(U_\pi) = \pi(\hat{S}(\omega)) = \pi(\hat{S}(\omega)^*)^* \\ &= (\iota\otimes\hat{S}(\omega)^*)(U_\pi)^* = (\iota\otimes\omega)(U_\pi^*).\end{aligned}$$

Thus $U_\pi^{-1} = U_\pi^*$, so U_π is unitary.

Therefore $\mathscr{A}$ is spanned by matrix coefficients of finite dimensional unitary corepresentations. By Theorem 1.6.7 we conclude that $(\mathscr{A},\Delta) = (\mathbb{C}[G],\Delta)$ for a compact quantum group G. Clearly, the functor $F\colon \mathscr{C} \to \operatorname{Rep} G$ defined by $F(\pi) = U_\pi$, $F(T) = T$ on morphisms and with $F_2 = \iota$, is a unitary tensor functor. It is also clear that $\operatorname{Mor}(\pi,\vartheta) = \operatorname{Mor}(U_\pi,U_\vartheta)$ for any π and ϑ in $\mathscr{C}$. Choosing representatives π_α of the equivalence classes of irreducible representations in $\mathscr{C}$ we conclude that the representations U_{π_α} of G are irreducible, pairwise nonequivalent, and their matrix coefficients span $\mathbb{C}[G]$. Hence these representations exhaust the equivalence classes of irreducible representations of G. It follows that F is a unitary monoidal equivalence. □

Note once again that Theorem 1.6.7 is more than what is needed in the above proof, since Lemma 2.3.12 and the assumption that $\mathscr{C}$ has conjugates make a significant part of the proof of Theorem 1.6.7 redundant.

References. — [**14**], [**20**], [**21**], [**23**], [**32**], [**43**], [**69**], [**74**], [**76**], [**81**], [**88**], [**96**].

2.4. DRINFELD-JIMBO DEFORMATION OF COMPACT LIE GROUPS

Let G be a simply connected semisimple compact Lie group and $\mathfrak{g}$ be its complexified Lie algebra. Fix a nondegenerate symmetric ad-invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$ such that its restriction to the real Lie algebra of G is negative definite. The standard normalization of this form requires that when restricting this form to a Cartan subalgebra and considering the dual form, we get $(\alpha,\alpha) = 2$ for every short root in every simple factor of $\mathfrak{g}$, but we do not assume this unless stated otherwise. From this data we will now define, for every $q > 0$, $q \neq 1$, a compact quantum group G_q. For $q = 1$ we put $G_1 = G$.

First we will define a deformation of the universal enveloping algebra $U\mathfrak{g}$. In order to fix notation recall briefly the structure of $U\mathfrak{g}$. This is a Hopf $*$-algebra with involution such that the real Lie algebra of G consists of skew-adjoint elements, and with comultiplication defined by

$$\hat{\Delta}(X) = X \otimes 1 + 1 \otimes X \quad \text{for} \quad X \in \mathfrak{g}.$$

Fix a maximal torus $T \subset G$, and denote by $\mathfrak{h} \subset \mathfrak{g}$ its complexified Lie algebra. For every root $\alpha \in \Delta \subset \mathfrak{h}^*$ put $d_\alpha = (\alpha, \alpha)/2$. Let $H_\alpha \in \mathfrak{h}$ be the element corresponding to the coroot $\alpha^\vee = \alpha/d_\alpha$ under the identification $\mathfrak{h} \cong \mathfrak{h}^*$, so that $(H_\alpha, H) = \alpha^\vee(H)$ for $H \in \mathfrak{h}$. Under the same identification let $h_\beta \in \mathfrak{h}$ be the element corresponding to $\beta \in \mathfrak{h}^*$, so $h_\alpha = d_\alpha H_\alpha$ for $\alpha \in \Delta$. Fix a system of simple roots $\{\alpha_1, \dots, \alpha_r\}$. For every positive root $\alpha \in \Delta_+$ choose

$$E_\alpha \in \mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

such that $(E_\alpha, E_\alpha^*) = d_\alpha^{-1}$, and put $F_\alpha = E_\alpha^* \in \mathfrak{g}_{-\alpha}$; then $[E_\alpha, F_\alpha] = H_\alpha$. For the simple roots $\alpha_1, \dots, \alpha_r$ we will use the subindices i instead of α_i. Then the elements E_i, F_i, H_i, $1 \le i \le r$, generate $\mathfrak{g}$ as a Lie algebra. Finally, recall that the Cartan matrix is defined by $a_{ij} = (\alpha_i^\vee, \alpha_j) = (\alpha_i, \alpha_j)/d_i$. We now define a new Hopf $*$-algebra by deforming the relations satisfied by the elements $E_i, F_i, q^{d_i H_i}$.

Definition 2.4.1. — For $q > 0$, $q \neq 1$, the quantized universal enveloping algebra $U_q\mathfrak{g}$ is defined as the universal unital algebra generated by elements E_i, F_i, K_i, K_i^{-1}, $1 \le i \le r$, satisfying the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0,$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \dfrac{[m]_{q_i}!}{[k]_{q_i}![m-k]_{q_i}!}$, $[m]_{q_i}! = [m]_{q_i}[m-1]_{q_i}\dots[1]_{q_i}$, $[n]_{q_i} = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $q_i = q^{d_i}$.

This is a Hopf $*$-algebra with coproduct $\hat\Delta_q$ defined by

$$\hat\Delta_q(K_i) = K_i \otimes K_i,\ \ \hat\Delta_q(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \ \hat\Delta_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

and with involution given by $K_i^* = K_i$, $E_i^* = F_iK_i$, $F_i^* = K_i^{-1}E_i$.

Note that the counit $\hat\varepsilon_q$ and the antipode $\hat S_q$ are given by

$$\hat\varepsilon_q(K_i) = 1,\ \ \hat\varepsilon_q(E_i) = \hat\varepsilon_q(F_i) = 0,$$

$$\hat S_q(K_i) = K_i^{-1},\ \ \hat S_q(E_i) = -K_i^{-1}E_i,\ \ \hat S_q(F_i) = -F_iK_i.$$

Lemma 2.4.2. — *The Hopf $*$-algebras $U_{q^{-1}}\mathfrak{g}$ and $U_q\mathfrak{g}$ are isomorphic.*

Proof. — In order to distinguish between the generators of $U_{q^{-1}}\mathfrak{g}$ and $U_q\mathfrak{g}$ we will use the superindices q^{-1} and q. We can then define an isomorphism by

$$E_i^{q^{-1}} \mapsto q_iK_i^qF_i^q = q_i^{-1}E_i^{q*},\ \ F_i^{q^{-1}} \mapsto q_i(K_i^q)^{-1}F_i^q = q_iF_i^{q*},\ \ K_i^{q^{-1}} \mapsto K_i^q. \qquad \square$$

We will next define a class of representations of $U_q\mathfrak{g}$. First let us introduce more notation. Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights, so $\omega_i(H_j) = \delta_{ij}$. We denote by Q and P the root and weight lattices, respectively. For a weight λ denote by $\lambda(i)$ the coefficients of λ in the basis $\omega_1, \dots, \omega_r$, so $\lambda(i) = \lambda(H_i) = (\lambda, \alpha_i)/d_i$. If V is a $U_q\mathfrak{g}$-module and $\lambda \in P$ is an integral weight, denote by $V(\lambda) \subset V$ the subspace of vectors of weight λ, defined by

$$V(\lambda) = \{\xi \in V \mid K_i\xi = q_i^{\lambda(i)}\xi = q^{(\lambda,\alpha_i)}\xi \text{ for all } i\}.$$

Definition 2.4.3. — A $U_q\mathfrak{g}$-module V is called admissible, or of type $\mathbf{1}$, if

$$V = \oplus_{\lambda\in P}V(\lambda).$$

Denote by $\mathscr{C}_q(\mathfrak{g})$ the C*-category of finite dimensional admissible unitary $U_q\mathfrak{g}$-modules.

Clearly, $\mathscr{C}_q(\mathfrak{g})$ is a C*-tensor category. In order to check that it has conjugates, let us compute $\hat S_q^2$. For an element $\alpha = \sum_i n_i\alpha_i$ of the root lattice put $K_\alpha = K_1^{n_1} \dots K_r^{n_r}$. We then have

$$K_\alpha E_i K_\alpha^{-1} = q^{(\alpha,\alpha_i)}E_i,\ \ K_\alpha F_i K_\alpha^{-1} = q^{-(\alpha,\alpha_i)}F_i.$$

Consider the weight $\rho = \omega_1 + \dots + \omega_r$. It is known that ρ also equals half the sum of the positive roots, so $2\rho \in Q$.

Lemma 2.4.4. — *For any $\omega \in U_q\mathfrak{g}$ we have $\hat S_q^2(\omega) = K_{2\rho}^{-1}\omega K_{2\rho}$.*

Proof. — This is immediate from

$$\hat S_q^2(K_i) = K_i,\ \ \hat S_q^2(E_i) = K_i^{-1}E_iK_i = q_i^{-a_{ii}}E_i = q^{-2d_i}E_i,\ \ \hat S_q^2(F_i) = q^{2d_i}F_i,$$

as $(2\rho, \alpha_i) = 2d_i\rho(i) = 2d_i$. $\square$

We thus see that in every finite dimensional admissible unitary $U_q\mathfrak{g}$-module V the square of the antipode is implemented by a positive invertible operator, the image of $K_{-2\rho}$. By Lemma 2.3.12 it follows that the contragredient module is unitarizable and its unitarization gives a conjugate object to V in $\operatorname{Rep} U_q\mathfrak{g}$. The contragredient module is clearly admissible. Therefore $\mathscr{C}_q(\mathfrak{g})$ has conjugates. Hence, by Theorem 2.3.13, it defines a compact quantum group.

Definition 2.4.5. — The Drinfeld-Jimbo q-deformation of G is the compact quantum group G_q with Hopf $*$-algebra $(\mathbb{C}[G_q],\Delta_q)$ defined as the subspace of the dual space $(U_q\mathfrak{g})^*$ spanned by the matrix coefficients of all finite dimensional admissible unitary modules, and with multiplication, involution and comultiplication defined by

$$(ab)(\omega)=(a\otimes b)\hat{\Delta}_q(\omega),\ \ a^*(\omega)=\overline{a(\hat{S}_q(\omega)^*)},\ \ \Delta_q(a)(\omega\otimes\nu)=a(\omega\nu).$$

Note that a priori the Hopf $*$-algebra $(\mathbb{C}[G_q],\Delta_q)$ does not completely determine $C(G_q)$, as it can have different C*-completions. But as we will see in Section 2.7, in the present case we have only one completion to a compact quantum group. Until then we can define $C(G_q)$ to be, for example, the C*-envelope of $\mathbb{C}[G_q]$. Note also that by Lemma 2.4.2 the quantum groups $G_{q^{-1}}$ and G_q are isomorphic.

By Theorem 2.3.13 we have a canonical unitary monoidal equivalences of categories $F\colon \mathscr{C}_q(\mathfrak{g}) \to \operatorname{Rep} G_q$. By this equivalence, if U is a finite dimensional unitary representation of G_q, then the $*$-representation $\pi_U\colon U_q\mathfrak{g} \to B(H_U)$ defined by $\pi_U(\omega) = (\iota\otimes\omega)(U)$ is equivalent to an admissible representation, hence it is itself admissible. It follows that F is not just an equivalence, but an isomorphism of categories. We can therefore identify $\mathscr{C}_q(\mathfrak{g})$ with $\operatorname{Rep} G_q$.

Remark 2.4.6. — In addition to admissible modules $U_q\mathfrak{g}$ has one-dimensional modules defined by $E_i\mapsto 0$, $F_i\mapsto 0$, $K_i\mapsto \pm1$. It can be shown, see [**18**, Section 10.1], that any finite dimensional module decomposes into a direct sum of admissible ones tensored with these one-dimensional modules. It follows that the category $\operatorname{Rep} U_q\mathfrak{g}$ of all finite dimensional unitary $U_q\mathfrak{g}$-modules is still a C*-tensor category with conjugates, and the corresponding compact quantum group is what can be denoted by $G_q \rtimes (\mathbb{Z}/2\mathbb{Z})^r$, where on the level of $U_q\mathfrak{g}$ the action of $(\mathbb{Z}/2\mathbb{Z})^r$ is defined as follows: the element 1 in the i-th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by mapping E_i to $-E_i$, F_i to $-F_i$, and by leaving all other generators intact. It also follows that $\mathscr{C}_q(\mathfrak{g})$ can be defined as the category of finite dimensional unitary $U_q\mathfrak{g}$-modules such that the elements K_i act by positive operators.

We will next describe, without proof, the structure of the category $\mathscr{C}_q(\mathfrak{g})$. It is very similar to the classical case.

A vector ξ in a $U_q\mathfrak{g}$-module $V\neq 0$ is called a highest weight vector of weight $\lambda\in P_+$ if $\xi\in V(\lambda)$, $E_i\xi=0$ for all i, and $V=(U_q\mathfrak{g})\xi$. If such a vector exists, then λ is uniquely determined and V is called a highest weight module. Clearly, any such module is admissible. The same argument as for $U\mathfrak{g}$ shows that for every $\lambda \in P_+$ there exists a unique irreducible highest weight module V_λ of weight λ.

Theorem 2.4.7. — *We have:*

(i) *for every $\lambda \in P_+$ the module V_λ is finite dimensional and unitarizable;*

(ii) *any finite dimensional admissible $U_q\mathfrak{g}$-module decomposes into a direct sum of the modules V_λ; in particular, any such module is completely reducible and unitarizable;*

(iii) *the dimensions of V_λ, as well as the dimensions of the weight spaces $V_\lambda(\mu)$, are the same as in the classical case;*

(iv) *the multiplicities $m^\eta_{\lambda,\nu}$ in the decompositions*

$$V_\lambda \otimes V_\nu \cong \bigoplus_{\eta \in P_+} \underbrace{V_\eta \oplus \cdots \oplus V_\eta}_{m^\eta_{\lambda,\nu}}$$

are the same as in the classical case.

Proofs of all these statements can be found for example in [**18**, Section 10.1].

From now on we fix a Hilbert space structure on V_λ making it a unitary module. When necessary, we will denote by π_λ the representation of $U_q\mathfrak{g}$ on V_λ. Fix also a unit vector $\xi_\lambda \in V_\lambda(\lambda)$.

As a consequence of complete reducibility, if V is a finite dimensional admissible $U_q\mathfrak{g}$-module and $\xi \in V(\lambda)$ is such that $E_i\xi = 0$ for all i, then there exists a unique morphism $V_\lambda \to V$ mapping ξ_λ into ξ. In particular, V_λ embeds into $V_{\omega_1}^{\otimes\lambda(1)} \otimes \cdots \otimes V_{\omega_r}^{\otimes\lambda(r)}$. It follows that $\mathbb{C}[G_q]$ is algebraically generated by the matrix coefficients of the fundamental modules V_{ω_i}; in fact, for all simple groups except the spin groups it suffices to take one particular fundamental module, see the discussion in [**75**]. It is, however, not easy to work out a complete list of relations. To the best of our knowledge this has been done only for the groups $SU(n)$ and $Sp(n)$, see [**51**].

Example 2.4.8. — Consider the group $G = SU(2)$ and the standardly normalized invariant form on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Let us show that for $0 < q < 1$ the quantum group G_q is exactly the quantum group $SU_q(2)$ introduced in Example 1.1.5.

The algebra $U_q\mathfrak{g}$ in this case is generated by elements E, F, K, K^{-1} satisfying the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The weight lattice is identified with $\frac{1}{2}\mathbb{Z}$. The modules V_s for nonnegative half-integers s can be explicitly written as follows. There exists an orthonormal basis $\{\xi^s_s, \xi^s_{s-1}, \dots, \xi^s_{-s}\}$

in V_s such that

$$\begin{aligned}E\xi_i^s &= q^{i+1}([s-i]_q[s+i+1]_q)^{1/2}\xi_{i+1}^s,\\ F\xi_i^s &= q^{-i}([s+i]_q[s-i+1]_q)^{1/2}\xi_{i-1}^s,\\ K\xi_i^s &= q^{2i}\xi_i^s;\end{aligned}$$

recall that $[n]_q = (q^n - q^{-n})/(q-q^{-1})$. In particular, the fundamental representation $\pi_{1/2}$ of $U_q\mathfrak{g}$, written in matrix form with respect to the basis $\{\xi_{1/2}^{1/2}, \xi_{-1/2}^{1/2}\}$, is

$$K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0\end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0\end{pmatrix}.$$

The contragredient representation $\pi_{1/2}^c$ has the form

$$K \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q\end{pmatrix}, \quad E \mapsto -\begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0\end{pmatrix}, \quad F \mapsto -\begin{pmatrix} 0 & q^{1/2} \\ 0 & 0\end{pmatrix}$$

with respect to the dual basis; recall that in matrix form we have $\pi_{1/2}^c(\omega) = \pi_{1/2}(\hat{S}_q(\omega))^t$. We see that $\begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0\end{pmatrix}$ intertwines $\pi_{1/2}^c$ with $\pi_{1/2}$. It follows that if $U = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}[G_q])$ is the unitary representation of G_q corresponding to $\pi_{1/2}$, then

$$\begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0\end{pmatrix}\begin{pmatrix} a^* & b^* \\ c^* & d^*\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0\end{pmatrix}.$$

Hence $d = a^*$ and $b = -qc^*$. Therefore we can define a surjective Hopf $*$-algebra homomorphism $\pi\colon \mathbb{C}[SU_q(2)] \to \mathbb{C}[G_q]$ by $\pi(\alpha) = a$ and $\pi(\gamma) = c$.

In order to prove that it is injective, consider the subspace $P_n \subset \mathbb{C}[SU_q(2)]$ of polynomials in α, α^*, γ, γ^* of degree n. Any such polynomial can be written as a sum of monomials of the form $\alpha^i\gamma^j(\gamma^*)^l$ and $(\alpha^*)^i\gamma^j(\gamma^*)^l$ with $i+j+l \le n$. There are exactly $(k+1)^2$ such monomials of degree k. Therefore $\dim P_n \le \sum_{k=0}^n (k+1)^2$.

On the other hand, since $V_{k/2}$ embeds into $V_{1/2}^{\otimes k}$, the space $\pi(P_n)$ contains the matrix coefficients of the simple modules $V_{k/2}$ for all $k \le n$. Hence

$$\dim \pi(P_n) \ge \sum_{k=0}^n (\dim V_{k/2})^2 = \sum_{k=0}^n (k+1)^2.$$

It follows that π is injective on P_n for all n, so it is injective on the whole algebra $\mathbb{C}[SU_q(2)]$. $\square$

Note that if the invariant form is not standardly normalized, then $SU(2)_q \cong SU_{q^s}(2)$ for some $s \in \mathbb{R}^*$. For simple groups it makes little sense not to normalize the invariant form. The main point in not doing this in the semisimple case, is that this way our definition of the q-deformation captures such quantum groups as e.g., $SU_q(2) \times SU_{q^s}(2)$.

Example 2.4.9. — Consider the group $G = SU(n)$ and the standardly normalized invariant form on $\mathfrak{g}$. The quantum group G_q is denoted by $SU_q(n)$. In this case to deduce relations in the algebra one uses two properties of the fundamental representation U. One is that its n-th tensor power contains a trivial subrepresentation. This plays the role of the determinant condition. The other one is that there is an explicit endomorphism of $U \times U$ that is a deformation of the flip; this will be discussed in Section 2.6. Together with counting arguments similar to the case $n = 2$ one then gets the following description of $\mathbb{C}[SU_q(n)]$ [**26, 83**]. It is a universal unital $*$-algebra generated by elements u_{ij}, $1 \le i, j \le n$, satisfying the relations

$$u_{ik}u_{jk} = qu_{jk}u_{ik}, \ \ u_{ki}u_{kj} = qu_{kj}u_{ki} \ \text{ for } i < j,$$

$$u_{il}u_{jk} = u_{jk}u_{il} \ \text{ for } i < j, \ k < l,$$

$$u_{ik}u_{jl} - u_{jl}u_{ik} = (q - q^{-1})u_{jk}u_{il} \ \text{ for } i < j, \ k < l,$$

$$\det_q(U) = 1,$$

$$u_{ij}^* = (-q)^{j-i} \det_q(U^{\hat{i}}_{\hat{j}}),$$

where $U = (u_{ij})_{i,j}$ and $\det_q(U) = \sum_{w \in S_n} (-q)^{\ell(w)} u_{w(1)1} \dots u_{w(n)n}$, with $\ell(w)$ being the number of inversions in $w \in S_n$, and where $U^{\hat{i}}_{\hat{j}}$ is the matrix obtained from U by removing the i-th row and the j-th column. The comultiplication is given by $\Delta_q(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. □

Although the C*-tensor category $\operatorname{Rep} G_q$ looks quite similar to $\operatorname{Rep} G$, these categories are not equivalent. One way to see this is to compare the intrinsic dimensions of simple objects. Note first that by definition we have a canonical unital $*$-homomorphism $U_q\mathfrak{g} \to \mathscr{U}(G_q) = \mathbb{C}[G_q]^*$ with weakly* dense image. We continue to denote by $\hat{\Delta}_q$ the 'coproduct' $\mathscr{U}(G_q) \to \mathscr{U}(G_q \times G_q)$.

Proposition 2.4.10. — *The Woronowicz character f_1 for G_q is equal to the image of $K_{-2\rho}$ in $\mathscr{U}(G_q)$. In particular,* $\dim_q V_\lambda = \operatorname{Tr} \pi_\lambda(K_{-2\rho}) = \operatorname{Tr} \pi_\lambda(K_{2\rho})$ *for any $\lambda \in P_+$.*

Proof. — By Remark 1.7.7 the element f_1 is completely characterized by the properties that it is positive, invertible, it implements the square of the antipode, and $\operatorname{Tr} \pi_\lambda(f_1) = \operatorname{Tr} \pi_\lambda(f_1^{-1})$ for all $\lambda \in P_+$. We already know that the element $K_{-2\rho}$ satisfies all these properties except the last one. This property, in turn, follows from the fact that the set of weights of V_λ is invariant under the Weyl group action, and $w_0\rho = -\rho$, where w_0 is the longest element in the Weyl group. Another possibility is to argue as follows. Since both f_1 and $K_{-2\rho}$ implement the square of the antipode, the image of $K_{-2\rho}$ is equal to zf_1 for some central element $z \in \mathscr{U}(G_q)$. Furthermore, z is positive and group-like, that is, it is invertible and $\hat{\Delta}_q(z) = z \otimes z$, since both $K_{-2\rho}$ and f_1 have these properties. As we will see at the end of Section 3.1, the unit is the only element with such properties. □

It is one of our goals to understand the exact relation between the representation categories of G_q and G.

As a consequence of the above proposition we see that the unitary antipode $\hat{R}_q$ on $\mathscr{U}(G_q)$ is given by

$$\hat{R}_q(\omega) = K_{2\rho}^{1/2} \hat{S}_q(\omega) K_{2\rho}^{-1/2}. \tag{2.4.1}$$

Note that the element $K_{2\rho}^{1/2}$ makes sense in $\mathscr{U}(G_q)$, but not in $U_q\mathfrak{g}$. Nevertheless the unitary antipode is well-defined on $U_q\mathfrak{g}$:

$$\hat{R}_q(K_i) = K_i^{-1},\ \ \hat{R}_q(E_i) = -q_i K_i^{-1} E_i,\ \ \hat{R}_q(F_i) = -q_i^{-1} F_i K_i. \tag{2.4.2}$$

One topic which we are not going to touch at all is the structure of the C*-algebras $C(G_q)$. It can be shown that these C*-algebras are of type I, their irreducible representations can be completely classified [**53**] and using this one can get very detailed information about the structure of $C(G_q)$ [**72**].

References. — [**18**], [**26**], [**20**], [**51**], [**53**], [**72**], [**75**], [**83**], [**96**].

2.5. REPRESENTATION CATEGORY OF $SU_q(2)$

In Example 2.4.8 we described all irreducible finite dimensional admissible $U_q\mathfrak{sl}_2$-modules for $q > 0$. It is also not difficult to understand how their tensor products decompose into irreducibles. Namely, using the notation of that example, consider the tensor product $V_s \otimes V_{1/2}$ for some $s \in \frac{1}{2}\mathbb{N}$. Then the vector $\xi^s_s \otimes \xi^{1/2}_{1/2}$ has weight $s + 1/2$ and is killed by E, so there exists a unique embedding $V_{s+1/2} \hookrightarrow V_s \otimes V_{1/2}$ mapping $\xi^{s+1/2}_{s+1/2}$ into $\xi^s_s \otimes \xi^{1/2}_{1/2}$. For similar reasons there exists a unique embedding $V_{s-1/2} \hookrightarrow V_s \otimes V_{1/2}$ such that

$$\xi^{s-1/2}_{s-1/2} \mapsto [2s]_q^{1/2} \xi^s_s \otimes \xi^{1/2}_{-1/2} - q^{s+1/2} \xi^s_{s-1} \otimes \xi^{1/2}_{1/2}.$$

For dimension reasons this exhausts all irreducible submodules of $V_s \otimes V_{1/2}$, so

$$V_s \otimes V_{1/2} \cong V_{s+1/2} \oplus V_{s-1/2}.$$

In particular, for $s = 1/2$ we see that $V_{1/2}$ is self-conjugate, which was already used in Example 2.4.8. It is also easy to see that by induction we get

$$V_s \otimes V_t \cong V_{|s-t|} \oplus V_{|s-t|+1} \oplus \cdots \oplus V_{s+t}$$

for all $s,t\in\frac{1}{2}\mathbb{Z}_+$. Our goal in this section is to show that these rules essentially characterize the representation categories $\operatorname{Rep} SU_q(2)$.

We thus start with an arbitrary strict C*-tensor category $\mathscr{C}$ with conjugates. Assume U is a nonzero self-conjugate object in $\mathscr{C}$, so there exist $R,\bar R\colon \mathbb{1}\to U\otimes U$ solving the conjugate equations. If in addition U is irreducible, then $\bar R=\lambda R$ for some $\lambda\in\mathbb{C}^*$. Then the conjugate equations read as

$$\bar\lambda(R^*\otimes\iota)(\iota\otimes R)=\iota,\quad \lambda(R^*\otimes\iota)(\iota\otimes R)=\iota,$$

so λ is real. Replacing R by $|\lambda|^{1/2}R$ and $\bar R$ by $|\lambda|^{-1/2}\bar R$, we may therefore assume that $\bar R=-\tau R$, with $\tau=1$ or $\tau=-1$. Let us assume that this is the case even if U is not irreducible. To exclude completely trivial cases let us also assume that $\|R\|>1$ (otherwise $d_i(U)=1$, so U is irreducible and $U\otimes U\cong\mathbb{1}$). Thus we assume that we have an object U and a morphism $R\colon\mathbb{1}\to U\otimes U$ such that

$$\|R\|^2=d\ \text{ and }\ -\tau(R^*\otimes\iota)(\iota\otimes R)=\iota, \tag{2.5.1}$$

with $d>1$ and $\tau=1$ or $\tau=-1$.

We want to understand the structure of morphisms that can be obtained from R and the identity morphisms by taking tensor products, adjoints and compositions. It is convenient to think of all these morphisms as living in one large algebra. Namely, consider the space $\oplus_{n,m\ge0}\operatorname{Mor}(U^{\otimes n},U^{\otimes m})$. Composition of morphisms and involution define a structure of a $*$-algebra on this space. This algebra has a unique completion to a C*-algebra, since for every $N\in\mathbb{N}$ the finite dimensional $*$-algebra $\oplus_{n,m=0}^N\operatorname{Mor}(U^{\otimes n},U^{\otimes m})$ can be identified with the C*-algebra $\operatorname{End}(\oplus_{k=0}^N U^{\otimes k})$. Thus we can say that our goal is to understand the C*-subalgebra generated by morphisms of the form

$$\underbrace{\iota\otimes\cdots\otimes\iota}_{i}\otimes R\otimes\underbrace{\iota\otimes\cdots\otimes\iota}_{j}\colon U^{\otimes(i+j)}\to U^{\otimes(i+j+2)}. \tag{2.5.2}$$

Let us try to axiomatize properties of these morphisms.

For a real number $d>1$ and a number $\tau=\pm1$, consider the universal C*-algebra $A_{d,\tau}$ generated by mutually orthogonal projections z_n, $n\ge0$, and partial isometries v_{ij}, $i,j\ge0$, such that

$$\begin{aligned}
v_{ij}^*v_{ij}&=z_{i+j},\quad v_{ij}v_{ij}^*\le z_{i+j+2},\\
v_{i,j+1}^*v_{i+1,j}&=-\tau d^{-1}z_{i+j+1},\\
v_{i,j+k+2}v_{i+j,k}&=v_{i+j+2,k}v_{i,j+k},\\
v_{i,j+k+2}^*v_{i+j+2,k}&=v_{i+j,k}v_{i,j+k}^*.
\end{aligned}$$

Denote by $A_{d,\tau}(n,m)$ the subspace $z_mA_{d,\tau}z_n$ of $A_{d,\tau}$.

Lemma 2.5.1. — *For every $n \geq 0$, the C*-algebra $A_{d,\tau}(n,n)$ is generated, as a unital C*-algebra with unit z_n, by the projections $e_{ni} = v_{i,n-i-2}v^*_{i,n-i-2}$, $0 \leq i \leq n-2$. These projections satisfy the relations*

$$e_{ni}e_{nj} = e_{nj}e_{ni} \ \text{ if } |i-j| \geq 2, \ \ e_{ni}e_{nj}e_{ni} = d^{-2}e_{ni} \ \text{ if } |i-j| = 1.$$

Proof. — First of all notice that the relations in $A_{d,\tau}$ imply that any element in the $*$-algebra generated by z_m and v_{ij} can be written as a linear combination of projections z_m and elements of the form

$$v_{i_1j_1}\dots v_{i_kj_k}v^*_{p_1q_1}\dots v^*_{p_lq_l}. \tag{2.5.3}$$

An element of the form (2.5.3) lies in $A_{d,\tau}(n',n'')$ for some n' and n'', and if it is nonzero and $n' = n''$, then $k = l$. It therefore suffices to show that any element of the form (2.5.3) with $k = l$ lies in the algebra $A^0_{d,\tau}$ generated by the projections z_m and e_{ij}. We will show this by induction on k.

For $k = 1$ consider an element $v_{ij}v^*_{pq}$. We may assume that $i+j = p+q$, as otherwise this element is zero. By taking, if necessary, the adjoint of this element we may also assume that $p \geq i$. Then

$$v_{ij}v^*_{pq} = (-\tau d)^{p-i}v_{ij}(v^*_{ij}v_{i+1,j-1})\dots(v^*_{p-1,q+1}v_{pq})v^*_{pq} \in A^0_{d,\tau}.$$

Assume the result is true for $k-1$. In order to prove it for k, it suffices to show that $v_{ij}A^0_{d,\tau}v^*_{pq} \subset A^0_{d,\tau}$. For this, in turn, it suffices to show that $v_{ij}(v_{st}v^*_{st})v^*_{pq} \in A^0_{d,\tau}$. Since $v_{st}v^*_{st} = v^*_{s,t+2}v_{s+2,t}$, this is true by the case $k = 1$.

The relations for e_{ni} are verified by a straightforward computation. For example, we have

$$\begin{aligned} e_{ni}e_{n,i+1}e_{ni} &= v_{i,n-i-2}(v^*_{i,n-i-2}v_{i+1,n-i-3})(v^*_{i+1,n-i-3}v_{i,n-i-2})v^*_{i,n-i-2} \\ &= d^{-2}v_{i,n-i-2}v^*_{i,n-i-2} = d^{-2}e_{ni}. \end{aligned}$$ □

The relations satisfied by the projections e_{ni}, called the Temperley-Lieb relations, are well studied, see e.g., [**36**]. In particular, it is known that unless $d \geq 2$ or $d = 2\cos(\pi/k)$ for an integer $k \geq 4$, for all sufficiently large n there exist no nonzero projections e_{ni}, $0 \leq i \leq n-2$, satisfying these relations, so $v_{ij} = 0$ for all i,j such that $i+j$ is large enough, and hence $A_{d,\tau} = 0$. We will briefly explain how to show that $A_{d,\tau} = 0$ for $d \in (1,2)$, $d \neq 2\cos(\pi/k)$, a bit later.

Therefore the algebra $A_{d,\tau}$ is of interest to us only if $d \geq 2$ or $d = 2\cos(\pi/k)$ $(k \geq 4)$. For such d it can be shown that $A_{d,\tau}$ is indeed nonzero. Namely, by construction $A_{d,\tau} \neq 0$ if there exists a C*-tensor category with a morphism $R\colon \mathbb{1} \to U \otimes U$ satisfying (2.5.1), since we then have a $*$-homomorphism from $A_{d,\tau}$ into the completion of $\oplus_{n,m\geq 0}\operatorname{Mor}(U^{\otimes n}, U^{\otimes m})$ mapping $d^{1/2}v_{ij}$ into the morphism $U^{\otimes(i+j)} \to U^{\otimes(i+j+2)}$

defined by (2.5.2), and z_n into the unit of $\mathrm{End}(U^{\otimes n})$. For $d \geq 2$ such morphisms R are easy to find in Hilb_f.

Example 2.5.2. — For $n \geq 2$ and $H = \mathbb{C}^n$, identify $\bar{H}$ with H and consider the canonical solution (r, r) of the conjugate equations for (H, H), so $r(1) = \sum_i e_i \otimes e_i$. By Proposition 2.2.5 any other solution has the form

$$R = (1 \otimes F)r, \quad \bar{R} = ((F^*)^{-1} \otimes 1)r = (1 \otimes \bar{F}^{-1})r$$

for some $F \in \mathrm{GL}_n(\mathbb{C})$; recall that we denote by $\bar{F}$ the matrix obtained from F by taking the complex conjugate of every entry. This solution satisfies $\bar{R} = -\tau R$ if and only if $F\bar{F} = -\tau 1$. We also have $\|R\|^2 = \mathrm{Tr}(F^*F)$. Therefore any matrix $F \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Tr}(F^*F) = d$ and $F\bar{F} = -\tau 1$ defines a morphism $R_F = (1 \otimes F)r$ satisfying (2.5.1), and this way we get all solutions of (2.5.1) in Hilb_f, for Hilbert spaces of dimension at least 2, up to unitary isomorphisms.

Taking $F = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$, with $\lambda^2 \in [-1, 1] \setminus \{0\}$, we get $d = |\lambda^2 + \lambda^{-2}|$ and $\tau = \mathrm{sign}(\lambda^2)$, so already for such matrices we obtain all possible combinations of $d \geq 2$ and $\tau = \pm 1$. ☐

On the other hand, for $d = 2\cos(\pi/k)$ it is clearly impossible to find a morphism R satisfying (2.5.1) in Hilb_f, since for Hilbert spaces relations (2.5.1) imply $\dim U \leq d < 2$ and therefore U must be one-dimensional and $d = 1$. Nevertheless the required morphisms can be found in some other C*-tensor categories [**15, 93**], but since such categories cannot have unitary fiber functors and therefore cannot lead to compact quantum groups, we will be less concerned with this case.

Fix now $d \geq 2$ or $d = 2\cos(\pi/k)$ $(k \geq 4)$, and $\tau = \pm 1$. We construct a C*-tensor category $\mathcal{TL}_{d,\tau}$ as follows. We start with a set of objects indexed by integers $n \geq 0$. Write $\underline{n}$ for the object corresponding to n. Put $\mathrm{Mor}(\underline{n}, \underline{m}) = A_{d,\tau}(n, m)$. The $*$-algebra structure on $A_{d,\tau}$ defines composition of morphisms and involution. Next define tensor product by $\underline{n} \otimes \underline{m} = \underline{n + m}$. The tensor product of morphisms is completely determined by the rules $v_{ij} \otimes z_n = v_{i,j+n}$, $z_n \otimes v_{ij} = v_{n+i,j}$. This way we get a category that satisfies all the properties of a strict C*-tensor category except that conditions (vi) (existence of direct sums) and (vii) (existence of subobjects) in Definition 2.1.1 do not hold. It is, however, not difficult to complete the category to get these conditions satisfied.

Assume $\mathcal{C}$ is a category having all the properties of a strict C*-tensor category except existence of direct sums and subobjects. First complete it with respect to direct sums. For this, consider the category $\mathcal{C}'$ consisting of n-tuples $(U_1, \ldots, U_n)$ of objects in $\mathcal{C}$

for all $n \geq 1$. Morphisms are defined by

$$\mathrm{Mor}((U_1,\dots,U_n),(V_1,\dots,V_m)) = \oplus_{i,j}\mathrm{Mor}(U_i,V_j).$$

Composition and involution are defined in the obvious way. Note that the norm is uniquely determined by the C*-condition $\|T\|^2 = \|T^*T\|$, and for its existence we do need condition (ii)(c) in Definition 2.1.1 to be satisfied in $\mathscr{C}$. The tensor product of $(U_1,\dots,U_n)$ and $(V_1,\dots,V_m)$ is defined as the nm-tuple consisting of objects $U_i\otimes V_j$ ordered lexicographically. The category $\mathscr{C}'$ has direct sums: they can be defined by concatenation. We next complete $\mathscr{C}'$ with respect to subobjects. For this, consider the category $\mathscr{C}''$ consisting of pairs (U,p), where U is an object in $\mathscr{C}'$ and p is a projection in $\mathrm{End}(U)$. Morphisms are defined by

$$\mathrm{Mor}((U,p),(V,q)) = q\,\mathrm{Mor}(U,V)p,$$

and tensor products by $(U,p)\otimes(V,q) = (U\otimes V, p\otimes q)$. The category $\mathscr{C}''$ is called the idempotent completion, or the Karoubi envelope, of $\mathscr{C}'$. It is a strict C*-tensor category. It is not difficult to see that the construction of $\mathscr{C}''$ from $\mathscr{C}$ is universal in the sense that if $\mathscr{B}$ is a C*-tensor category and $F\colon \mathscr{C}\to\mathscr{B}$ is a unitary tensor functor, then F extends uniquely, up to a natural unitary monoidal isomorphism, to a unitary tensor functor $\mathscr{C}''\to\mathscr{B}$.

Applying the above procedure to our category with objects $\underline{n}$ and morphisms $\mathrm{Mor}(\underline{n},\underline{m}) = A_{d,\tau}(n,m)$, we get a strict C*-tensor category $\mathscr{TL}_{d,\tau}$. It is called the Temperley-Lieb category for $d \geq 2$ and the reduced Temperley-Lieb category for $d = 2\cos(\pi/k)$.

The unit $\mathbb{1}$ in $\mathscr{TL}_{d,\tau}$ is the object $\underline{0}$. The object $\underline{1}$ is simple, and by analogy with $SU_q(2)$ we denote it by $U_{1/2}$. The category $\mathscr{TL}_{d,\tau}$ is generated by $U_{1/2}$, in the sense that any simple object in $\mathscr{TL}_{d,\tau}$ is isomorphic to a subobject of $U_{1/2}^{\otimes n}$. Consider the morphism

$$R_{1/2} = d^{-1/2}v_{00}\colon \mathbb{1}\to U_{1/2}\otimes U_{1/2}.$$

Then the pair $(R_{1/2}, -\tau R_{1/2})$ solves the conjugate equations for $U_{1/2}$, so the simple object $U_{1/2}$ is self-conjugate and $d_i(U_{1/2}) = d$. Since $\mathscr{TL}_{d,\tau}$ is generated by an object that has a conjugate, it is a C*-tensor category with conjugates.

Given a C*-tensor category $\mathscr{C}$ and a unitary tensor functor $F\colon \mathscr{TL}_{d,\tau}\to\mathscr{C}$, consider the object $U = F(U_{1/2})$ and morphism $R = F_2^*F(R_{1/2})F_0\colon \mathbb{1}\to U\otimes U$. Then R satisfies (2.5.1), that is,

$$\|R\|^2 = d \ \text{ and } \ -\tau(R^*\otimes\iota)(\iota\otimes R) = \iota.$$

Conversely, starting from these relations we can construct a unitary tensor functor.

Theorem 2.5.3. — *Assume $\mathscr{C}$ is a strict C*-tensor category, U is an object in $\mathscr{C}$, and $R\colon \mathbb{1}\to U\otimes U$ is a morphism satisfying* (2.5.1) *for some $d > 1$ and $\tau = \pm 1$. Then $d \geq 2$ or $d =$*

$2\cos(\pi/k)$ $(k \geq 4)$*, and there exists a unique, up to a natural unitary monoidal isomorphism, unitary tensor functor* $F\colon \mathcal{TL}_{d,\tau} \to \mathcal{C}$ *such that* $F(U_{1/2}) = U$ *and* $F(R_{1/2}) = F_2RF_0^*$*. If* (U', R') *is another such pair in* $\mathcal{C}$*, with the same* d *and* τ*, and* $F'\colon \mathcal{TL}_{d,\tau} \to \mathcal{C}$ *is the corresponding functor, then the functors* F *and* F' *are naturally unitarily monoidally isomorphic if and only if there exists a unitary* $T\colon U \to U'$ *such that* $R' = (T \otimes T)R$*.*

Proof. — As we already discussed before Example 2.5.2, given (U, R) satisfying (2.5.1), we can define a $*$-homomorphism π from $A_{d,\tau}$ into the completion of $\oplus_{n,m\geq 0} \operatorname{Mor}(U^{\otimes n}, U^{\otimes m})$. From this we conclude that $d \geq 2$ or $d = 2\cos(\pi/k)$ $(k \geq 4)$, since otherwise $A_{d,\tau} = 0$ and we would get $\operatorname{End}(U^{\otimes n}) = 0$ for all $n \geq 0$. To define a unitary tensor functor F, put

$$F(U_{1/2}^{\otimes n}) = U^{\otimes n} \text{ for } n \geq 0, \quad F(T) = \pi(T) \text{ for } T \in \operatorname{Mor}(U_{1/2}^{\otimes n}, U_{1/2}^{\otimes m}).$$

We also define $F_0\colon \mathbb{1} \to \mathbb{1}$ and $F_2(U_{1/2}^{\otimes n}, U_{1/2}^{\otimes m})\colon U^{\otimes n} \otimes U^{\otimes m} \to U^{\otimes(n+m)}$ to be the identity maps. Since $\mathcal{TL}_{d,\tau}$ is the completion of the category consisting of objects $U_{1/2}^{\otimes n}$ with respect to direct sums and subobjects, these formulas define a unique, up to a natural unitary monoidal isomorphism, unitary tensor functor.

Assume now that we have another unitary tensor functor $\tilde{F}\colon \mathcal{TL}_{d,\tau} \to \mathcal{C}$ such that $\tilde{F}(U_{1/2}) = U$ and $\tilde{F}(R_{1/2}) = \tilde{F}_2R\tilde{F}_0^*$. The tensor structure $\tilde{F}_2$ of $\tilde{F}$ defines unitaries $\eta_{U_{1/2}^{\otimes n}}\colon \tilde{F}(U_{1/2})^{\otimes n} \to \tilde{F}(U_{1/2}^{\otimes n})$ such that

$$\tilde{F}_2(U_{1/2}^{\otimes n}, U_{1/2}^{\otimes m})(\eta_{U_{1/2}^{\otimes n}} \otimes \eta_{U_{1/2}^{\otimes m}}) = \eta_{U_{1/2}^{\otimes(n+m)}}$$

for all $n, m \geq 0$, with $\eta_{\mathbb{1}} = \tilde{F}_0$ and $\eta_{U_{1/2}} = \iota$. It is easy to check that

$$\eta_{U_{1/2}^{\otimes(i+j+2)}} F(v_{ij}) = \tilde{F}(v_{ij}) \eta_{U_{1/2}^{\otimes(i+j)}}.$$

This implies that the unitaries $\eta_{U_{1/2}^{\otimes n}}$ define a natural unitary monoidal isomorphism η between the functors F and $\tilde{F}$ restricted to the subcategory consisting of objects $U_{1/2}^{\otimes n}$, $n \geq 0$. Since $\mathcal{TL}_{d,\tau}$ is the completion of this category with respect to direct sums and subobjects, η extends uniquely to a natural unitary monoidal isomorphism between F and $\tilde{F}$.

For the second part of the theorem we have to show that if $F, F'\colon \mathcal{TL}_{d,\tau} \to \mathcal{C}$ are two unitary tensor functors, $U = F(U_{1/2})$, $U' = F'(U_{1/2})$, $R = F_2^*F(R_{1/2})F_0$ and $R' = F_2'^*F'(R_{1/2})F_0'$, then F and F' are naturally unitarily monoidally isomorphic if an only if there exists a unitary $T\colon U \to U'$ such that $R' = (T \otimes T)R$. In one direction this is obvious. Namely, if $\eta\colon F \to F'$ is a natural unitary monoidal isomorphism, then we can take $T = \eta_{U_{1/2}}$.

Conversely, assume we have a unitary $T\colon U \to U'$ such that $R' = (T \otimes T)R$. By the proof of the first part of the theorem we may assume that

$$F(U_{1/2}^{\otimes n}) = U^{\otimes n}, \quad F_0 = \iota, \quad F_2(U_{1/2}^{\otimes n}, U_{1/2}^{\otimes m}) = \iota,$$

and similarly for F'. Then the unitaries $T^{\otimes n}\colon F(U_{1/2}^{\otimes n}) \to F'(U_{1/2}^{\otimes n})$ define a natural unitary monoidal isomorphism between the functors F and F' restricted to the subcategory consisting of objects $U_{1/2}^{\otimes n}$, $n \geq 0$. This isomorphism extends uniquely to a natural unitary monoidal isomorphism between F and F'. □

Let us apply this theorem to $\mathscr{C} = \mathrm{Hilb}_f$. For $d = 2\cos(\pi/k)$ we already know that there are no morphisms in Hilb_f satisfying (2.5.1), hence no unitary fiber functors $\mathscr{TL}_{d,\tau} \to \mathrm{Hilb}_f$. On the other hand, for $d \geq 2$ in Example 2.5.2 we described all solutions of (2.5.1) in Hilb_f up to unitary isomorphisms. Namely, every matrix $F \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Tr}(F^*F) = d$ and $F\bar{F} = -\tau 1$ defines such a solution R_F. Denote by φ_F a unitary fiber functor $\mathscr{TL}_{d,\tau} \to \mathrm{Hilb}_f$ such that $\varphi_F(U_{1/2}^{\otimes m}) = (\mathbb{C}^n)^{\otimes m}$, $\varphi_F(R_{1/2}) = R_F$ and $\varphi_{F,2}(U_{1/2}^{\otimes k}, U_{1/2}^{\otimes m}) = \iota$.

Corollary 2.5.4. — *For any $d \geq 2$ and $\tau = \pm 1$, we have:*

(i) *any unitary fiber functor $\mathscr{TL}_{d,\tau} \to \mathrm{Hilb}_f$ is naturally unitarily monoidally isomorphic to φ_F for some $F \in \mathrm{GL}_n(\mathbb{C})$ (such that $\mathrm{Tr}(F^*F) = d$ and $F\bar{F} = -\tau 1$);*

(i) *two unitary fiber functors φ_F and $\varphi_{F'}$ are naturally unitarily monoidally isomorphic if and only if the matrices F and F' have the same size and there exists a unitary matrix v such that $F' = vFv^t$.*

Proof. — By the above discussion we only have to prove part (ii). But this is immediate, since for a unitary matrix v we have

$$(v \otimes v)R_F = (v \otimes vF)r = (1 \otimes vFv^t)r,$$

and therefore the condition $R_{F'} = (v \otimes v)R_F$ is equivalent to $F' = vFv^t$. □

Remark 2.5.5. — It is not difficult to classify matrices F up to equivalence $F \sim vFv^t$. First of all notice that this is the same as classifying the anti-linear operators JF up to unitary conjugation, where $J\colon \mathbb{C}^n \to \mathbb{C}^n$ is the complex conjugation. The condition $F\bar{F} = -\tau 1$ implies that if $F = u|F|$ is the polar decomposition, then $(Ju)^2 = -\tau 1$ and $(Ju)|F|(Ju)^* = |F|^{-1}$. It follows that $\mathbb{C}^n$ decomposes into a direct sum $H_- \oplus H_0 \oplus H_+$ such that the restriction of $|F|$ to H_- has eigenvalues < 1, $JuH_- = H_+$ and $|F|Ju = Ju|F|^{-1}$ on H_-, $|F|$ acts on H_0 as the identity operator and Ju acts on H_0 as an anti-linear isometry with square $-\tau 1$. It is easy to see that such an anti-linear isometry on H_0 is unique up to a unitary conjugation (see the proof of Lemma 4.4.1), and for $\tau = 1$ it exists only if H_0 is even-dimensional. Hence the unitary conjugacy class of JF is completely determined by the eigenvalues of $|F|$ that are < 1 counted with multiplicities. To summarize, for fixed $d \geq 2$, $\tau = \pm 1$ and $n \geq 2$ there is a one-to-one correspondence between equivalence classes of $F \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Tr}(F^*F) = d$, $F\bar{F} = -\tau 1$ and, assuming that n is even if $\tau = 1$, vectors $(\lambda_1, \dots, \lambda_k)$, $0 \leq k \leq n/2$, such that $0 < \lambda_1 \leq \dots \leq \lambda_k < 1$ and $\sum_{i=1}^k (\lambda_i^2 + \lambda_i^{-2}) = d - (n - 2k)$. If $\tau = 1$ and n is odd, then no such F exists. In particular, if $n=2$, then for every $d\geq 2$ and $\tau=\pm 1$ we have exactly one equivalence class.

By Woronowicz's Tannaka-Krein duality, the functors φ_F define compact quantum groups. These are nothing else than the free orthogonal quantum groups $A_o(F)$ defined in Example 1.1.7.

Proposition 2.5.6. — *For $d\geq 2$ and $\tau=\pm 1$, assume $F\in \mathrm{GL}_n(\mathbb{C})$ is such that $\mathrm{Tr}(F^*F)=d$ and $F\bar F=-\tau 1$. Then there exists a unique, up to a natural unitary monoidal isomorphism, unitary monoidal equivalence $E\colon \mathcal{TL}_{d,\tau}\to \mathrm{Rep}\,A_o(F)$ such that $E(U_{1/2})=U$, where U is the fundamental representation of $A_o(F)$. The functor φ_F is naturally unitarily monoidally isomorphic to the composition of the canonical fiber functor $\mathrm{Rep}\,A_o(F)\to \mathrm{Hilb}_f$ with E.*

Proof. — By definition we have $F\in \mathrm{Mor}(U^c,U)$, whence $R_F=(1\otimes F)r\in \mathrm{Mor}(\mathbb{1},U\times U)$. It follows that there exists a unitary tensor functor $E\colon \mathcal{TL}_{d,\tau}\to \mathrm{Rep}\,A_o(F)$ such that $E(U_{1/2}^{\otimes m})=U^{\otimes m}$, $E(R_{1/2})=R_F$ and $E_2(U_{1/2}^{\otimes k},U_{1/2}^{\otimes m})=\iota$. On the other hand, by Woronowicz's Tannaka-Krein duality the functor φ_F can be thought of as a unitary monoidal equivalence between $\mathcal{TL}_{d,\tau}$ and $\mathrm{Rep}\,G$ for some compact quantum group G. Put $V=\varphi_F(U_{1/2})$. Then

$$R_F=(1\otimes F)r\in \mathrm{Mor}(\mathbb{1},V\times V),$$

or equivalently, $F\in \mathrm{Mor}(V^c,V)$. By universality of $A_o(F)$, it follows that there exists a unital $*$-homomorphism $(A_o(F),\Delta)\to (C(G),\Delta)$ mapping the matrix coefficients of U into those of V. This homomorphism defines a unitary tensor functor $\tilde E\colon \mathrm{Rep}\,A_o(F)\to \mathrm{Rep}\,G$. The functors φ_F and $\tilde E E$ are naturally unitarily monoidally isomorphic as functors $\mathcal{TL}_{d,\tau}\to \mathrm{Rep}\,G$. Since φ_F is an equivalence of categories, this implies that E is a unitary monoidal equivalence between $\mathcal{TL}_{d,\tau}$ and the full C*-tensor subcategory of $\mathrm{Rep}\,A_o(F)$ generated by U. But since the matrix coefficients of U generate $A_o(F)$ as a unital C*-algebra, this subcategory coincides with $\mathrm{Rep}\,A_o(F)$.

The uniqueness part of the proposition is equivalent to the statement that any unitary monoidal autoequivalence of $\mathcal{TL}_{d,\tau}$ that maps $U_{1/2}$ into itself is naturally unitarily monoidally isomorphic to the identity functor. Such an autoequivalence corresponds to a morphism $R\colon \mathbb{1}\to U_{1/2}\otimes U_{1/2}$ satisfying (2.5.1), and we have to show that $R=(T\otimes T)R_{1/2}$ for a unitary T. But this is obvious, since $U_{1/2}$ is irreducible and therefore R is a scalar multiple of $R_{1/2}$. □

Our next goal is to understand the structure of simple objects in $\mathcal{TL}_{d,\tau}$ for $d\geq 2$. For $\tau=1$ we already know it, since by the previous proposition $\mathcal{TL}_{d,1}$ is unitarily monoidally equivalent to $\mathrm{Rep}\,SU_q(2)$, where $q\in(0,1]$ is such that $d=q+q^{-1}$ (see Examples 1.1.7 and 2.5.2). But we will give an argument that works equally well for $\tau=-1$ and does not rely on quantized universal enveloping algebras.

Recall that in Section 2.2 we introduced traces Tr_U on $\mathrm{End}(U)$ for any C*-tensor category with conjugates. For the categories $\mathscr{TL}_{d,\tau}$, in order to simplify the notation, denote $\mathrm{Tr}_{U_{1/2}^{\otimes n}}$ by Tr_n. Recall also that z_n denotes the unit in $\mathrm{End}(U_{1/2}^{\otimes n}) = A_{d,\tau}(n,n)$. Therefore $\mathrm{Tr}_n(z_n) = d_i(U_{1/2}^{\otimes n}) = d^n$. We will write v for $v_{00} = d^{-1/2}R_{1/2}$ and e for $e_{2,0} = vv^*$. Then $v_{ij} = z_i \otimes v \otimes z_j$ and $e_{ni} = z_i \otimes e \otimes z_{n-i-2}$.

Lemma 2.5.7. — *Assume $d \geq 2$ or $d = 2\cos(\pi/k)$ $(k \geq 4)$, and $\tau = \pm 1$. Then for any $n \geq 2$ and any morphism $T \in \mathrm{End}(U_{1/2}^{\otimes(n-1)})$ in $\mathscr{TL}_{d,\tau}$, we have*

$$\mathrm{Tr}_n((T \otimes z_1)(z_{n-2} \otimes e)) = d^{-2}\,\mathrm{Tr}_n(T \otimes z_1) = d^{-1}\,\mathrm{Tr}_{n-1}(T).$$

This property of the trace is called the Markov property. In order to prove it, we will introduce partial traces.

Assume we are given a strict C*-tensor category $\mathscr{C}$ with conjugates. For every object U choose a standard solution $(R_U, \bar{R}_U)$ of the conjugate equations for U. Then define linear maps $\mathrm{Tr}_U \otimes \iota \colon \mathrm{End}(U \otimes V) \to \mathrm{End}(V)$ by

$$(\mathrm{Tr}_U \otimes \iota)(T) = (R_U^* \otimes \iota)(\iota \otimes T)(R_U \otimes \iota).$$

By Proposition 2.2.15 these maps are independent of the choice of a standard solution. Similarly, define $\iota \otimes \mathrm{Tr}_U \colon \mathrm{End}(V \otimes U) \to \mathrm{End}(V)$ by

$$(\iota \otimes \mathrm{Tr}_U)(T) = (\iota \otimes \bar{R}_U^*)(T \otimes \iota)(\iota \otimes \bar{R}_U).$$

Note that the morphisms $\mathrm{Tr}_U(\iota \otimes \mathrm{Tr}_V)$ and $\mathrm{Tr}_V(\mathrm{Tr}_U \otimes \iota)$ coincide and, by the computation in the proof of Theorem 2.2.18, are equal to $\mathrm{Tr}_{U \otimes V}$.

Proof of Lemma 2.5.7. — Since $\mathrm{Tr}_n = \mathrm{Tr}_{n-1}(\iota \otimes \mathrm{Tr}_1)$, we just have to show that $(\iota \otimes \mathrm{Tr}_1)(z_{n-2} \otimes e) = d^{-1} z_{n-1}$. Equivalently, we have to show that $(\iota \otimes \mathrm{Tr}_1)(e) = d^{-1} z_1$, or in other words,

$$(\iota \otimes \mathrm{Tr}_1)(R_{1/2}R_{1/2}^*) = \iota.$$

But this identity follows immediately from $-\tau(\iota \otimes R_{1/2}^*)(R_{1/2} \otimes \iota) = \iota$. □

For every $n \geq 2$ consider the subalgebra I_n of $A_{d,\tau}(n,n)$ generated by the projections e_{ni}, $0 \leq i \leq n-2$. It is an ideal, hence it has the form $(z_n - f_n)A_{d,\tau}(n,n)$ for a central projection f_n in $A_{d,\tau}(n,n)$. Since $A_{d,\tau}(n,n) = I_n + \mathbb{C}z_n$, this projection is either zero or minimal in $A_{d,\tau}(n,n)$. Put $f_1 = z_1$ and $f_0 = z_0$.

Lemma 2.5.8. — *Assume $d \geq 2$ and $\tau = \pm 1$. Let $q > 0$ be such that $d = q + q^{-1}$. Then in $A_{d,\tau}$ for every $n \geq 2$ we have:*

(i) *the projection f_n is nonzero and* $\mathrm{Tr}_n(f_n) = [n+1]_q$;

(ii) $f_n = f_{n-1} \otimes z_1 - d\dfrac{[n-1]_q}{[n]_q}(f_{n-1} \otimes z_1)(z_{n-2} \otimes e)(f_{n-1} \otimes z_1)$;

(iii) *the projection* $d\dfrac{[n-1]_q}{[n]_q}(f_{n-1}\otimes z_1)(z_{n-2}\otimes e)(f_{n-1}\otimes z_1)$ *is equivalent to* f_{n-2}.

Proof. — We will prove this by induction on n.

For $n=2$ we have $f_2=z_2-e$. Since $e=vv^*$, we have $\mathrm{Tr}_2(e)=\mathrm{Tr}_0(z_0)=1$, so $\mathrm{Tr}_2(f_2)=d^2-1=q^2+1+q^{-2}=[3]_q$. Properties (ii) and (iii) are obvious.

Assume now that the lemma is true for n. To prove it for $n+1$, we start with part (iii). Consider the elements

$$p=\frac{d[n]_q}{[n+1]_q}(f_n\otimes z_1)(z_{n-1}\otimes e)(f_n\otimes z_1),\ x=\left(\frac{d[n]_q}{[n+1]_q}\right)^{1/2}(f_n\otimes z_1)(z_{n-1}\otimes v).$$

Then $p=xx^*$. On the other hand,

$$x^*x=\frac{d[n]_q}{[n+1]_q}(z_{n-1}\otimes v^*)(f_n\otimes z_1)(z_{n-1}\otimes v).$$

Since

$$f_n\otimes z_1=f_{n-1}\otimes z_2-d\frac{[n-1]_q}{[n]_q}(f_{n-1}\otimes z_2)(z_{n-2}\otimes e\otimes z_1)(f_{n-1}\otimes z_2)$$

and

$$(z_1\otimes v^*)(e\otimes z_1)(z_1\otimes v)=d^{-2}(z_1\otimes v^*)(z_1\otimes e)(z_1\otimes v)=d^{-2}z_1,$$

we get

$$x^*x=\frac{d[n]_q}{[n+1]_q}\left(f_{n-1}-d^{-1}\frac{[n-1]_q}{[n]_q}f_{n-1}\right)=f_{n-1},$$

as $d[n]_q-[n-1]_q=[n+1]_q$. We see that p is indeed a projection, and p is equivalent to f_{n-1} in $A_{d,\tau}$.

Turning to (ii), we already know that p is a projection. Since $p\le f_n\otimes z_1$, it follows that $\tilde f_{n+1}=f_n\otimes z_1-p$ is a projection. Since f_n is orthogonal to e_{ni} for all $i\le n-2$, the projection $\tilde f_{n+1}$ is orthogonal to $e_{n+1,i}=e_{ni}\otimes z_1$ for all $i\le n-2$. We also have

$$\frac{d[n]_q}{[n+1]_q}(z_{n-1}\otimes e)(f_n\otimes z_1)(z_{n-1}\otimes e)=(z_{n-1}\otimes v)x^*x(z_{n-1}\otimes v^*)=f_{n-1}\otimes e,$$

and this implies that

$$p(z_{n-1}\otimes e)=(f_n\otimes z_1)(f_{n-1}\otimes e)=(f_n\otimes z_1)(z_{n-1}\otimes e),$$

since $f_n\le f_{n-1}\otimes z_1$. Therefore $\tilde f_{n+1}=f_n\otimes z_1-p$ is orthogonal to $z_{n-1}\otimes e=e_{n+1,n-1}$, hence $\tilde f_{n+1}\le f_{n+1}$. On the other hand, since $f_{n+1}\le f_n\otimes z_1$ and f_{n+1} is orthogonal to $z_{n-1}\otimes e$, we have

$$\tilde f_{n+1}f_{n+1}=(f_n\otimes z_1)f_{n+1}=f_{n+1}.$$

Hence $\tilde f_{n+1}=f_{n+1}$. This proves (ii).

Finally, using (ii) and Lemma 2.5.7 we compute:

$$\mathrm{Tr}_{n+1}(f_{n+1}) = d\,\mathrm{Tr}_n(f_n) - \frac{[n]_q}{[n+1]_q}\,\mathrm{Tr}_n(f_n) = d[n+1]_q - [n]_q = [n+2]_q,$$

which finishes the proof of the lemma for $n+1$. $\square$

For $d \geq 2$ and $\tau = \pm 1$, denote by $U_{n/2}$ ($n \geq 2$) a simple object in $\mathcal{TL}_{d,\tau}$ defined by the minimal projection $f_n \in \mathrm{End}(U_{1/2}^{\otimes n})$. Let us also write U_0 for $\mathbb{1}$. Note that by part (i) of the above lemma we have $d_i(U_s) = [2s+1]_q$ for all $s \in \frac{1}{2}\mathbb{Z}_+$.

Theorem 2.5.9. — *For any $d \geq 2$ and $\tau = \pm 1$, the simple objects U_s, $s \in \frac{1}{2}\mathbb{Z}_+$, in $\mathcal{TL}_{d,\tau}$ are pairwise nonisomorphic, and any simple object in $\mathcal{TL}_{d,\tau}$ is isomorphic to one of them. For any $s,t \in \frac{1}{2}\mathbb{Z}_+$ we have*

$$U_s \otimes U_t \cong U_{|s-t|} \oplus U_{|s-t|+1} \oplus \cdots \oplus U_{s+t}.$$

Proof. — The objects U_s, $s \in \frac{1}{2}\mathbb{Z}_+$, in $\mathcal{TL}_{d,\tau}$ are pairwise nonisomorphic simply because they have different intrinsic dimensions. Alternatively, by the proof of Lemma 2.5.1 we have $A_{d,\tau}(n-2,n) = \sum_{i=0}^{n-2} v_{i,n-i-2}A_{d,\tau}(n-2,n-2)$, there are no nonzero morphisms $U_{1/2}^{\otimes k} \to U_{1/2}^{\otimes n}$ for $k \not\equiv n \pmod 2$, and any morphism $U_{1/2}^{\otimes k} \to U_{1/2}^{\otimes n}$ for $k < n$ such that $k \equiv n \pmod 2$ factors through $U_{1/2}^{\otimes(n-2)}$. This implies that $f_nT = 0$ for any morphism $T\colon U_{1/2}^{\otimes k} \to U_{1/2}^{\otimes n}$ with $k < n$, so that $U_{n/2}$ cannot be isomorphic to $U_{k/2}$ for $k < n$.

By Lemma 2.5.8 we have $f_{n-1} \otimes z_1 = f_n + p_n$, where p_n is a projection equivalent to f_{n-2}. This implies that

$$U_{(n-1)/2} \otimes U_{1/2} \cong U_{n/2} \oplus U_{(n-2)/2} \ \text{ for } \ n \geq 2.$$

From this the claimed formula for $U_s \otimes U_t$, $t = n/2$, is obtained by a simple induction on n, using that

$$(U_s \otimes U_{(n+1)/2}) \oplus (U_s \otimes U_{(n-1)/2}) \cong (U_s \otimes U_{n/2}) \otimes U_{1/2} \ \text{ for } \ n \geq 1.$$

It remains to show that any simple object in $\mathcal{TL}_{d,\tau}$ is isomorphic to one of U_s. For this it suffices to show that the tensor powers $U_{1/2}^{\otimes n}$ decompose into multiple copies of U_s. But this is again easy to check by induction on n. $\square$

The above arguments, with small modifications, work also for $d = 2\cos(\pi/k)$. The main difference is that eventually the projections f_n become equal to zero. These arguments also show where the restrictions on $d < 2$ come from.

Briefly, assume $d \in (1,2)$ is such that $A_{d,\tau} \neq 0$, so that we can define the category $\mathcal{TL}_{d,\tau}$. Let $c > 3$ be such that $d = 2\cos(\pi/c)$. Put $q = e^{\pi i/c}$, so that $d = [2]_q$. The induction argument in the proof of Lemma 2.5.8 works as long as the numbers $[n+1]_q = \frac{\sin(\pi(n+1)/c)}{\sin(\pi/c)}$ remain strictly positive. Let $k \geq 4$ be the integer such that $k-1 < c \leq k$. We then get that $\mathrm{Tr}_{k-1}(f_{k-1}) = [k]_q$. If $c \neq k$, then $\mathrm{Tr}_{k-1}(f_{k-1}) < 0$, so we get a contradiction. On the other hand, if $c = k$, then we conclude that $f_n = 0$ for all $n \geq k-1$. In this case the category $\mathcal{TL}_{d,\tau}$ contains only $k-1$ nonisomorphic simple objects $U_{n/2}$, $0 \leq n \leq k-2$,

$$U_{n/2} \otimes U_{1/2} \cong U_{(n+1)/2} \oplus U_{(n-1)/2} \text{ for } 1 \leq n \leq k-3,$$

and $U_{(k-2)/2} \otimes U_{1/2} \cong U_{(k-3)/2}$. We leave the details to the reader.

Theorem 2.5.9 characterizes the categories $\mathcal{TL}_{d,\tau}$ in the following sense.

Theorem 2.5.10. — *Assume $\mathscr{C}$ is a C^*-tensor category with conjugates such that its set of isomorphism classes of simple objects has representatives V_s, $s \in \frac{1}{2}\mathbb{Z}_+$, such that*

$$V_s \otimes V_t \cong V_{|s-t|} \oplus V_{|s-t|+1} \oplus \cdots \oplus V_{s+t}.$$

Then $\mathscr{C}$ is unitarily monoidally equivalent to $\mathcal{TL}_{d,\tau}$ for uniquely defined $d \geq 2$ and $\tau = \pm 1$.

Proof. — We may assume that $\mathscr{C}$ is strict. The unit in $\mathscr{C}$ must be isomorphic to V_0. It follows that $\mathbb{1}$ is a subobject of $V_{1/2} \otimes V_{1/2}$, hence $V_{1/2}$ is self-conjugate. Put $d = d_i(V_{1/2})$, and τ to be the sign of $-(R^* \otimes \iota)(\iota \otimes R)$, where R is a nonzero morphism $\mathbb{1} \to V_{1/2} \otimes V_{1/2}$ (recall the discussion at the beginning of the section). Then by Theorem 2.5.3 there exists a unitary tensor functor $F\colon \mathcal{TL}_{d,\tau} \to \mathscr{C}$ such that $F(U_{1/2}) = V_{1/2}$. Comparing the decompositions of $U_{n/2} \otimes U_{1/2}$ and $V_{n/2} \otimes V_{1/2}$ into simple objects, an easy induction argument shows that $F(U_{n/2}) \cong V_{n/2}$ for all n if $d \geq 2$, and $F(U_{n/2}) \cong V_{n/2}$ for $n \leq k-2$ and $V_{(k-1)/2}$ is a zero object for $d = 2\cos(\pi/k)$. In the latter case we get a contradiction, as $V_{(k-1)/2}$ was assumed to be simple. Hence $d \geq 2$ and F is a unitary monoidal equivalence of categories.

It remains to show that d and τ are uniquely determined. The object $U_{1/2}$ in $\mathcal{TL}_{d,\tau}$ ($d \geq 2$) can, for example, be characterized as follows: it is a unique, up to an isomorphism, object in $\mathcal{TL}_{d,\tau}$ with minimal intrinsic dimension among simple objects that are nonisomorphic to the unit object. This uniquely recovers d and τ from $\mathcal{TL}_{d,\tau}$. ◻

Returning to quantum groups, the results of this section can be summarized as follows.

Theorem 2.5.11. — *We have:*

(i) *every free orthogonal quantum group $A_o(F)$ is monoidally equivalent to $SU_q(2)$ for a uniquely defined $q \in [-1,1] \setminus \{0\}$; namely, for q such that* $\mathrm{Tr}(F^*F) = |q+q^{-1}|$ *and* $\mathrm{sign}(F\bar{F}) = -\mathrm{sign}(q)$*;*

(ii) *two free orthogonal quantum groups $A_o(F)$ and $A_o(F')$ are isomorphic if and only if the matrices F and F' have the same size and there exists a unitary matrix v such that $F' = vFv^t$;*

(iii) *if G is a compact quantum group with representation category as in the formulation of Theorem 2.5.10, then the universal form of G is isomorphic to $A_o(F)$ for some F.*

Proof. — (i) By Proposition 2.5.6 the representation category of $A_o(F)$ is unitarily monoidally equivalent to $\mathcal{TL}_{d,\tau}$, where $d = \mathrm{Tr}(F^*F)$ and $\tau = -\operatorname{sign}(F\bar{F})$. By Theorem 2.5.10 the numbers d and τ are uniquely determined by the category. By Example 1.1.7, describing $SU_q(2)$ as a free orthogonal quantum group, this gives the result.

(ii) An isomorphism of compact quantum groups $A_o(F)$ and $A_o(F')$ defines a unitary monoidal equivalence of their representation categories that intertwines their canonical fiber functors. We can identify these categories with $\mathcal{TL}_{d,\tau}$ for some $d \geq 2$ and $\tau = \pm 1$. We then get a unitary monoidal autoequivalence of $\mathcal{TL}_{d,\tau}$ intertwining the fiber functors φ_F and $\varphi_{F'}$ that define $A_o(F)$ and $A_o(F')$. Such an autoequivalence must map $U_{1/2}$ into an isomorphic object by the proof of Theorem 2.5.10. As was shown in the proof of Proposition 2.5.6, it follows that the autoequivalence is naturally unitarily monoidally isomorphic to the identity functor. Therefore the fiber functors φ_F and $\varphi_{F'}$ are naturally unitarily monoidally isomorphic. By Corollary 2.5.4 this implies that the matrices F and F' have the same size and there exists a unitary matrix v such that $F' = vFv^t$.

The converse follows easily by definition of free orthogonal quantum groups, as well as from Corollary 2.5.4, since naturally unitarily monoidally isomorphic unitary fiber functors define isomorphic Hopf $*$-algebras.

(iii) By Theorem 2.5.10 the representation category of G is unitarily monoidally equivalent to $\mathcal{TL}_{d,\tau}$ for some $d \geq 2$ and $\tau = \pm 1$. The canonical fiber functor $\operatorname{Rep} G \to \mathrm{Hilb}_f$ defines a unitary fiber functor $\mathcal{TL}_{d,\tau} \to \mathrm{Hilb}_f$. By Corollary 2.5.4 this functor is naturally unitarily monoidally isomorphic to φ_F for some F. Hence the Hopf $*$-algebra of matrix coefficients of finite dimensional representations of G is isomorphic to that of $A_o(F)$. It follows that $(C_u(G), \Delta) \cong (A_o(F), \Delta)$. □

Finishing this section, we note that the representation categories of most of the compact quantum groups we have encountered so far can be characterized by some intrinsic properties, see the references.

References. — [**3**], [**4**], [**7**], [**13**], [**14**], [**15**], [**22**], [**36**], [**50**], [**73**], [**74**], [**79**], [**80**], [**91**], [**93**], [**98**], [**99**].

2.6. BRAIDED AND RIBBON CATEGORIES

If G is a compact group, then for any representations U and V of G the flip map $H_U \otimes H_V \to H_V \otimes H_U$ defines an equivalence between $U \times V$ and $V \times U$. This is no longer the case for quantum groups, when $\hat{\Delta}$ is not cocommutative. Nevertheless, it can happen that the representations $U \times V$ and $V \times U$ are still equivalent in a natural way. This leads to the following definition.

Definition 2.6.1. — A braiding on a C*-tensor category $\mathscr{C}$ is a collection of natural isomorphisms $\sigma_{U,V}: U \otimes V \to V \otimes U$ such that the hexagon diagram

$$\begin{tikzcd}
(V \otimes U) \otimes W \arrow[d, "\alpha"'] & (U \otimes V) \otimes W \arrow[l, "\sigma\otimes\iota"'] \arrow[r, "\alpha"] & U \otimes (V \otimes W) \arrow[d, "\sigma_{1,23}"] \\
V \otimes (U \otimes W) \arrow[r, "\iota\otimes\sigma"] & V \otimes (W \otimes U) & (V \otimes W) \otimes U \arrow[l, "\alpha"']
\end{tikzcd}$$

and the same diagram with σ replaced by σ^{-1} both commute.

When a braiding is fixed, the category $\mathscr{C}$ is called braided. If in addition $\sigma^2 = \iota$, then $\mathscr{C}$ is called symmetric.

A monoidal equivalence F between braided C*-tensor categories $\mathscr{C}$ and $\mathscr{C}'$ is called braided if the diagram

$$\begin{tikzcd}
F(U) \otimes F(V) \arrow[r, "F_2"] \arrow[d, "\sigma'"'] & F(U \otimes V) \arrow[d, "F(\sigma)"] \\
F(V) \otimes F(U) \arrow[r, "F_2"] & F(V \otimes U)
\end{tikzcd}$$

commutes.

It may seem natural to require the braiding to be unitary. In our main examples, however, it will be self-adjoint, that is, $\sigma_{U,V} = \sigma^*_{V,U}$.

We will mainly be interested in braidings on the representation category $\operatorname{Rep} G$ of a compact quantum group G. Consider the forgetful functor $F\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, that is, $F(U) = H_U$ on objects and $F(T) = T$ on morphisms. Then σ followed by the flip $\Sigma\colon H_V \otimes H_U \to H_U \otimes H_V$ defines a natural transformation $\mathscr{R}$ from $F^{\otimes 2}$ into itself. As we have already used in Section 2.3, the algebra $\operatorname{End}(F^{\otimes n})$ can be identified with $\mathscr{U}(G^n)$, so $\mathscr{R} \in \mathscr{U}(G \times G)$. More concretely, $\mathscr{R}$ is the unique element in $\mathscr{U}(G \times G)$ such that

$$\sigma_{U,V} = \Sigma(\pi_U \otimes \pi_V)(\mathscr{R}).$$

We will often simply write $\sigma = \Sigma\mathscr{R}$.

It easy to check that the defining properties of σ translate into the following properties of $\mathscr{R}$.

Definition 2.6.2. — An R-matrix for G is an invertible element $\mathscr{R} \in \mathscr{U}(G \times G)$ such that

(i) $\mathscr{R}\hat\Delta(\cdot)\mathscr{R}^{-1} = \hat\Delta^{\mathrm{op}}$,

(ii) $(\iota \otimes \hat\Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$ and $(\hat\Delta \otimes \iota)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$.

Thus we have a one-to-one correspondence between braidings on $\operatorname{Rep} G$ and R-matrices.

Note that the condition $\sigma^* = \sigma$ is equivalent to $\mathscr{R}^* = \mathscr{R}_{21}$, while the condition $\sigma^2 = \iota$ is equivalent to $\mathscr{R}^{-1} = \mathscr{R}_{21}$.

Example 2.6.3. — Let G be the dual of a discrete abelian group Γ. Then $\mathscr{U}(G \times G)$ is the algebra of functions on $\Gamma\times\Gamma$, so an invertible element in $\mathscr{U}(G\times G)$ is a function $\Gamma\times\Gamma \to \mathbb{C}^*$. Since $\hat{\Delta}^{\mathrm{op}} = \hat{\Delta}$, condition (i) is satisfied for any such function $\mathscr{R}$. Condition (ii) means that $\mathscr{R}$ is a bi-quasi-character, that is, $\mathscr{R}(a+b,c) = \mathscr{R}(a,c)\mathscr{R}(b,c)$ and $\mathscr{R}(a,b+c) = \mathscr{R}(a,b)\mathscr{R}(a,c)$ for all $a,b,c \in \Gamma$. □

Example 2.6.4. — Consider the q-deformation G_q of a simply connected semisimple compact Lie group. An element of $\mathscr{U}(G_q \times G_q)$ is determined by its action on the modules $\bar{V}_\lambda \otimes V_\mu$, where $\bar{V}_\lambda$ denotes the conjugate to V_λ, so $\omega\bar{\xi} = \overline{\hat{R}_q(\omega)^*\xi}$. It can be shown that there exists an R-matrix $\mathscr{R}_q$ such that

$$\mathscr{R}_q(\bar{\xi}_\lambda \otimes \xi_\mu) = q^{-(\lambda,\mu)}\bar{\xi}_\lambda \otimes \xi_\mu. \tag{2.6.1}$$

This completely determines $\mathscr{R}_q$, since the vector $\bar{\xi}_\lambda \otimes \xi_\mu$ is cyclic, as can be easily checked using that ξ_μ is a highest weight vector and $\bar{\xi}_\lambda$ is a lowest weight vector, so that $F_i\bar{\xi}_\lambda = 0$. The R-matrix $\mathscr{R}_q$ has the form

$$\mathscr{R}_q = q^{\sum_{i,j}(B^{-1})_{ij}H_i\otimes H_j} \prod_{\alpha\in\Delta_+} \exp_{q_\alpha}((1-q_\alpha^{-2})F_\alpha \otimes E_\alpha), \tag{2.6.2}$$

where B is the matrix $((\alpha_i^\vee,\alpha_j^\vee))_{i,j}$, H_i is the self-adjoint element in $\mathscr{U}(G_q)$ such that $K_i = q^{d_iH_i}$, the product is taken with respect to a particular order on the set Δ_+ of positive roots, $q_\alpha = q^{d_\alpha}$,

$$\exp_q(\omega) = \sum_{n=0}^{\infty} q^{n(n+1)/2}\frac{\omega^n}{[n]_q!},$$

and E_α (resp. F_α) is a polynomial in $K_i^{\pm1}$ and E_i (resp. F_i) such that $K_iE_\alpha = q^{(\alpha_i,\alpha)}E_\alpha K_i$ (resp. $K_iF_\alpha = q^{-(\alpha_i,\alpha)}F_\alpha K_i$), see [**18**, Theorem 8.3.9] (in the conventions of [**18**] we have $q = e^h$, $K_i = e^{-d_ihH_i}$, $E_i = X_i^-$ and $F_i = X_i^+$).

Note that the series $\exp_{q_\alpha}((1-q_\alpha^{-2})F_\alpha \otimes E_\alpha)$ is convergent in the weak* topology on $\mathscr{U}(G_q \times G_q) = (\mathbb{C}[G_q] \otimes \mathbb{C}[G_q])^*$, since the elements E_α and F_α act by nilpotent operators on every finite dimensional admissible $U_q\mathfrak{g}$-module. Note also that (2.6.1) indeed holds, because $(F_\alpha \otimes E_\alpha)(\bar{\xi}_\lambda \otimes \xi_\mu) = 0$, $H_i\bar{\xi}_\lambda = -(\lambda,\alpha_i^\vee)\bar{\xi}_\lambda$, $H_j\xi_\mu = (\mu,\alpha_j^\vee)\xi_\mu$ and $\sum_{i,j}(B^{-1})_{i,j}(\lambda,\alpha_i^\vee)(\mu,\alpha_j^\vee) = (\lambda,\mu)$.

Finally, observe that the element $(\mathscr{R}_q)_{21}^*$ has the form

$$\left(\prod_{\alpha\in\Delta_+} \exp_{q_\alpha}((1-q_\alpha^{-2})\tilde{F}_\alpha \otimes \tilde{E}_\alpha)\right) q^{\sum_{i,j}(B^{-1})_{ij}H_i\otimes H_j},$$

where $\tilde{E}_\alpha = F_\alpha^*$ (resp. $\tilde{F}_\alpha = E_\alpha^*$) is again a polynomial in $K_i^{\pm 1}$ and E_i (resp. F_i) such that $K_i\tilde{E}_\alpha = q^{(\alpha_i,\alpha)}\tilde{E}_\alpha K_i$ (resp. $K_i\tilde{F}_\alpha = q^{-(\alpha_i,\alpha)}\tilde{F}_\alpha K_i$). It follows that $(\mathscr{R}_q)_{21}^*(\bar{\xi}_\lambda\otimes\xi_\mu) = q^{-(\lambda,\mu)}\bar{\xi}_\lambda\otimes\xi_\mu$, hence $(\mathscr{R}_q)_{21}^* = \mathscr{R}_q$.

From now on we consider the category $\operatorname{Rep} G_q = \mathscr{C}_q(\mathfrak{g})$ as a braided C*-tensor category with braiding $\Sigma\mathscr{R}_q$. □

Let us establish a number of general properties of braidings and R-matrices.

Theorem 2.6.5. — *For any braiding on a strict C*-tensor category we have:*

(i) $(\iota_W\otimes\sigma_{U,V})(\sigma_{U,W}\otimes\iota_V)(\iota_U\otimes\sigma_{V,W}) = (\sigma_{V,W}\otimes\iota_U)(\iota_V\otimes\sigma_{U,W})(\sigma_{U,V}\otimes\iota_W)$;

(ii) $\sigma_{\mathbb{1},U} = \sigma_{U,\mathbb{1}} = \iota$.

Proof. — We will only show (i), leaving (ii) to the reader. By the hexagon relations the left hand side in (i) equals $(\iota_W\otimes\sigma_{U,V})\sigma_{U\otimes V,W}$. By naturality this is equal to $\sigma_{V\otimes U,W}(\sigma_{U,V}\otimes\iota_W)$, which is exactly the right hand side in (i), again by the hexagon relations. □

Properties (i) and (ii) of R-matrices in the following theorem are a direct translation of the properties of braidings, but it is equally easy to deduce them from scratch.

Theorem 2.6.6. — *Assume $\mathscr{R}$ is an R-matrix for a compact quantum group G. Then*

(i) $\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}$ *(Yang-Baxter equation);*

(ii) $(\hat{\varepsilon}\otimes\iota)(\mathscr{R}) = 1 = (\iota\otimes\hat{\varepsilon})(\mathscr{R})$*;*

(iii) $(\hat{S}\otimes\iota)(\mathscr{R}) = \mathscr{R}^{-1}$ *and* $(\iota\otimes\hat{S})(\mathscr{R}^{-1}) = \mathscr{R}$*;*

(iv) $(\hat{S}\otimes\hat{S})(\mathscr{R}) = \mathscr{R} = (\hat{R}\otimes\hat{R})(\mathscr{R})$.

Proof. — (i) Since $(\hat{\Delta}\otimes\iota)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$, by applying the flip to the first two factors we also get $(\hat{\Delta}^{\mathrm{op}}\otimes\iota)(\mathscr{R}) = \mathscr{R}_{23}\mathscr{R}_{13}$. On the other hand,

$$(\hat{\Delta}^{\mathrm{op}}\otimes\iota)(\mathscr{R}) = \mathscr{R}_{12}(\hat{\Delta}\otimes\iota)(\mathscr{R})\mathscr{R}_{12}^{-1} = \mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23}\mathscr{R}_{12}^{-1},$$

hence $\mathscr{R}_{23}\mathscr{R}_{13} = \mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23}\mathscr{R}_{12}^{-1}$.

(ii) By applying $\hat{\varepsilon}\otimes\iota\otimes\iota$ to $(\hat{\Delta}\otimes\iota)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$ we get $\mathscr{R} = (\hat{\varepsilon}\otimes\iota)(\mathscr{R})\mathscr{R}$, whence $(\hat{\varepsilon}\otimes\iota)(\mathscr{R}) = 1$. Similarly, using $(\iota\otimes\hat{\Delta})(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$ we get $(\iota\otimes\hat{\varepsilon})(\mathscr{R}) = 1$.

(iii) As in the case of corepresentations in Section 1.6, applying $m(\iota\otimes\hat{S})\otimes\iota$ to $(\hat{\Delta}\otimes\iota)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$ we get $1 = \mathscr{R}(\hat{S}\otimes\iota)(\mathscr{R})$. Similarly, applying $\iota\otimes m(\hat{S}\otimes\iota)$ to $(\iota\otimes\hat{\Delta})(\mathscr{R}^{-1}) = \mathscr{R}_{12}^{-1}\mathscr{R}_{13}^{-1}$ we get $1 = (\iota\otimes\hat{S})(\mathscr{R}^{-1})\mathscr{R}^{-1}$.

(iv) The identity $(\hat{S}\otimes\hat{S})(\mathscr{R}) = \mathscr{R}$ follows immediately from (iii). Since $\hat{S} = (\operatorname{Ad}\rho^{1/2})\hat{R}$, it follows that

$$(\rho^{1/2}\otimes\rho^{1/2})(\hat{R}\otimes\hat{R})(\mathscr{R})(\rho^{-1/2}\otimes\rho^{-1/2}) = \mathscr{R}.$$

But $\mathscr{R}$ commutes with $(\rho^{1/2}\otimes\rho^{1/2})$, since

$$\mathscr{R}(\rho^{1/2}\otimes\rho^{1/2}) = \mathscr{R}\hat{\Delta}(\rho^{1/2}) = \hat{\Delta}^{\mathrm{op}}(\rho^{1/2})\mathscr{R} = (\rho^{1/2}\otimes\rho^{1/2})\mathscr{R}.$$

Hence $(\hat{R}\otimes\hat{R})(\mathscr{R})=\mathscr{R}$. □

Our next goal is to introduce a so called ribbon structure on braided C*-tensor categories with conjugates. This will lead to some interesting identities for braidings, but overall this notion will play only a peripheral role for us. It is, however, an important notion in more general categories than C*-categories. We refer the reader to [**48**] for a thorough discussion and for an explanation of the term 'ribbon'. One nice property of ribbon categories is that they allow for a notion of dimension of an object. In particular, for braided C*-tensor categories with conjugates we will see a relation between the intrinsic dimension and its more algebraic counterparts.

Definition 2.6.7. — A twist on a braided C*-tensor category $\mathscr{C}$ with conjugates is a collection θ of natural isomorphisms $\theta_U\colon U\to U$ such that

$$\theta_{U\otimes V}=(\theta_U\otimes\theta_V)\sigma_{V,U}\sigma_{U,V}\ \text{ and }\ \theta_{\bar U}=(\theta_U)^\vee.$$

When a twist is fixed, $\mathscr{C}$ is called a ribbon C*-category.

Note that by naturality of θ and the proof of Lemma 2.3.3, the morphism $(\theta_U)^\vee$ does not depend on the choice of a solution of the conjugate equations for U and $\bar U$, so the second condition in the definition of a twist is meaningful. Furthermore, the proof of that lemma shows that for any collection θ of natural isomorphisms we can define a collection $\theta^\vee$ of natural isomorphisms such that $(\theta^\vee)_{\bar U}=(\theta_U)^\vee$ for any U with conjugate $\bar U$. Therefore the second condition on the twist can be written as $\theta^\vee=\theta$.

As we already mentioned, given a twist we can define dimension of an object. Namely, consider the scalar in $\operatorname{End}(\mathbb{1})=\mathbb{C}1$ defined as the composition

$$\mathbb{1}\xrightarrow{\bar R}U\otimes\bar U\xrightarrow{\theta_U\otimes\iota}U\otimes\bar U\xrightarrow{\sigma_{U,\bar U}}\bar U\otimes U\xrightarrow{R^*}\mathbb{1}.$$

Using Proposition 2.2.5 it is easy to check that it is independent of the choice of $\bar U$ and a solution $(R,\bar R)$ of the conjugate equations. This scalar is called the quantum dimension of U and is denoted by $\dim_q U$. In order to distinguish it from the quantum dimension on $\operatorname{Rep} G$ defined in Section 2.2 we will instead write $\dim_q^\theta U$. It can be shown that $\dim_q^\theta$ is multiplicative on tensor products, additive on direct sums, and that $\dim_q^\theta\bar U=\dim_q^\theta U$. The last property is immediate from the following lemma.

Lemma 2.6.8. — *Assume $\mathscr{C}$ is a strict braided C*-tensor category, and $(R,\bar R)$ is a solution of the conjugate equations for an object U. Then*

$$R^*\sigma_{U,\bar U}(T\otimes\iota)\bar R=\bar R^*\sigma_{\bar U,U}(T^\vee\otimes\iota)R\ \textit{ for all }\ T\in\operatorname{End}(U).$$

Proof. — Let us first show that

$$\sigma_{U,\bar U}=(\bar R^*\otimes\iota_{\bar U}\otimes\iota_U)(\iota_U\otimes\sigma^{-1}_{\bar U,\bar U}\otimes\iota_U)(\iota_U\otimes\iota_{\bar U}\otimes R).\tag{2.6.3}$$

By the hexagon identities we have $\iota_U\otimes\sigma^{-1}_{\bar U,\bar U}=\sigma^{-1}_{U\otimes\bar U,\bar U}(\sigma_{U,\bar U}\otimes\iota_{\bar U})$. Observe next that

$$(\bar R^*\otimes\iota_{\bar U})\sigma^{-1}_{U\otimes\bar U,\bar U}=\sigma^{-1}_{\mathbb{1},\bar U}(\iota_{\bar U}\otimes\bar R^*)=\iota_{\bar U}\otimes\bar R^*.$$

It follows that

$$(\bar R^*\otimes\iota_{\bar U})(\iota_U\otimes\sigma^{-1}_{\bar U,\bar U})=(\iota_{\bar U}\otimes\bar R^*)(\sigma_{U,\bar U}\otimes\iota_{\bar U}).$$

Hence the right hand side of (2.6.3) equals

$$(\iota_{\bar U}\otimes\bar R^*\otimes\iota_U)(\sigma_{U,\bar U}\otimes\iota_{\bar U}\otimes\iota_U)(\iota_U\otimes\iota_{\bar U}\otimes R)=\sigma_{U,\bar U}.$$

Similarly it is proved that

$$\sigma_{\bar U,U}=(\iota_U\otimes\iota_{\bar U}\otimes R^*)(\iota_U\otimes\sigma^{-1}_{\bar U,\bar U}\otimes\iota_U)(R\otimes\iota_{\bar U}\otimes\iota_U),\tag{2.6.4}$$

by first checking that $(\iota_{\bar U}\otimes R^*)(\sigma^{-1}_{\bar U,\bar U}\otimes\iota_U)=(R^*\otimes\iota_{\bar U})(\iota_{\bar U}\otimes\sigma_{\bar U,U})$.

Using identities (2.6.3) and (2.6.4) we now compute:

$$\begin{aligned}R^*\sigma_{U,\bar U}(T\otimes\iota)\bar R&=R^*(\bar R^*\otimes\iota\otimes\iota)(\iota\otimes\sigma^{-1}_{\bar U,\bar U}\otimes\iota)(\iota\otimes\iota\otimes R)(T\otimes\iota)\bar R\\&=(\bar R^*\otimes R^*)(T\otimes\sigma^{-1}_{\bar U,\bar U}\otimes\iota)(\bar R\otimes R)\\&=\bar R^*(T\otimes\iota)(\iota\otimes\iota\otimes R^*)(\iota\otimes\sigma^{-1}_{\bar U,\bar U}\otimes\iota)(\bar R\otimes\iota\otimes\iota)R\\&=\bar R^*(T\otimes\iota)\sigma_{\bar U,U}R=\bar R^*\sigma_{\bar U,U}(\iota\otimes T)R\\&=\bar R^*\sigma_{\bar U,U}(T^\vee\otimes\iota)R.\end{aligned}$$ □

Our goal is to show that every braided C*-tensor category with conjugates has a canonical ribbon structure. In order to formulate the result recall that in Section 2.5 we introduced partial traces $\mathrm{Tr}_U\otimes\iota$ and $\iota\otimes\mathrm{Tr}_V$ on $\mathrm{End}(U\otimes V)$.

Theorem 2.6.9. — *Let $\mathscr{C}$ be a strict braided C*-tensor category with conjugates. For every object U put $\theta_U=(\mathrm{Tr}_U\otimes\iota)(\sigma_{U,U})\in\mathrm{End}(U)$. Then $\theta_U=(\iota\otimes\mathrm{Tr}_U)(\sigma_{U,U})$, θ_U is invertible with inverse equal to $(\mathrm{Tr}_U\otimes\iota)(\sigma^{-1}_{U,U})=(\iota\otimes\mathrm{Tr}_U)(\sigma^{-1}_{U,U})$, the collection $\theta=(\theta_U)_U$ is a twist on $\mathscr{C}$, and $\dim^\theta_q$ coincides with the intrinsic dimension d_i. If in addition σ is self-adjoint, then θ is self-adjoint, and if σ is unitary, then θ is unitary.*

Proof. — We proceed in several steps.

Step 1. The maps θ_U are natural in U.

Assume $T\in\mathrm{Mor}(U,V)$. In order to show that $\theta_VT=T\theta_U$ it suffices to check that $\mathrm{Tr}_U(S\theta_VT)=\mathrm{Tr}_U(ST\theta_U)$ for all $S\in\mathrm{Mor}(V,U)$. We have

$$\mathrm{Tr}_U(S\theta_VT)=\mathrm{Tr}_U(\mathrm{Tr}_V\otimes\iota)((\iota\otimes S)\sigma_{U,U}(\iota\otimes T))=\mathrm{Tr}_{V\otimes U}((T\otimes S)\sigma_{V,U}),$$

and

$$\mathrm{Tr}_U(ST\theta_U)=\mathrm{Tr}_U(\mathrm{Tr}_U\otimes\iota)((\iota\otimes ST)\sigma_{U,U})=\mathrm{Tr}_{U\otimes U}((\iota\otimes S)\sigma_{V,U}(T\otimes\iota)).$$

The last expression equals $\mathrm{Tr}_{V\otimes U}((T\otimes S)\sigma_{V,U})$ by the tracial property of Tr. Hence $\theta_VT=T\theta_U$.

Step 2. We have $\theta_{U\otimes V} = (\theta_U \otimes \theta_V)\sigma_{V,U}\sigma_{U,V}$.

Using the hexagon identities we can write

$$\sigma_{U\otimes V,U\otimes V} = (\iota_U \otimes \sigma_{U,V} \otimes \iota_V)(\sigma_{U,U} \otimes \iota_V \otimes \iota_V)(\iota_U \otimes \iota_U \otimes \sigma_{V,V})(\iota_U \otimes \sigma_{V,U} \otimes \iota_V).$$

By the proof of Theorem 2.2.18 we have, with the obvious notation, $\mathrm{Tr}_{U\otimes V} \otimes\iota \otimes \iota = (\mathrm{Tr}_V \otimes\iota \otimes \iota)(\mathrm{Tr}_U \otimes\iota \otimes \iota \otimes \iota)$ on $\mathrm{End}(U \otimes V \otimes U \otimes V)$. Applying $\mathrm{Tr}_U \otimes\iota \otimes \iota \otimes \iota$ to the above identity we get

$$\begin{aligned}(\mathrm{Tr}_U \otimes\iota \otimes \iota \otimes \iota)(\sigma_{U\otimes V,U\otimes V}) &= (\sigma_{U,V} \otimes \iota_V)(\theta_U \otimes \iota_V \otimes \iota_V)(\iota_U \otimes \sigma_{V,V})(\sigma_{V,U} \otimes \iota_V)\\ &= (\iota_V \otimes \theta_U \otimes \iota_V)(\sigma_{U,V} \otimes \iota_V)(\iota_U \otimes \sigma_{V,V})(\sigma_{V,U} \otimes \iota_V)\\ &= (\iota_V \otimes \theta_U \otimes \iota_V)(\iota_V \otimes \sigma_{V,U})(\sigma_{V,V} \otimes \iota_U)(\iota_V \otimes \sigma_{U,V}),\end{aligned}$$

where in the last step we used Theorem 2.6.5(i). Applying $\mathrm{Tr}_V \otimes\iota \otimes \iota$ we then get

$$\theta_{U\otimes V} = (\theta_U \otimes \iota_V)\sigma_{V,U}(\theta_V \otimes \iota_U)\sigma_{U,V} = (\theta_U \otimes \theta_V)\sigma_{V,U}\sigma_{U,V}.$$

Step 3. We have $\sigma^{-1}_{U,\bar U}R_U = (\theta_U \otimes \iota)\bar R_U$ *and* $\theta_U^{-1} = (\iota \otimes \mathrm{Tr}_U)(\sigma^{-1}_{U,U})$, *where* $(R_U, \bar R_U)$ *is a standard solution of the conjugate equations for* U.

By the Frobenius reciprocity, the map

$$\mathrm{End}(U) \to \mathrm{Mor}(\mathbb{1}, U \otimes \bar U),\ \ T \mapsto (T \otimes \iota)\bar R_U,$$

is a linear isomorphism. Let $T \in \mathrm{End}(U)$ be such that $\sigma^{-1}_{U,\bar U}R_U = (T \otimes \iota)\bar R_U$. Then

$$\begin{aligned}T &= (\iota_U \otimes R_U^*)(\sigma^{-1}_{U,\bar U} \otimes \iota_U)(R_U \otimes \iota_U)\\ &= (\iota_U \otimes R_U^*)\sigma^{-1}_{U,\bar U\otimes U}(\iota_{\bar U} \otimes \sigma_{U,U})(R_U \otimes \iota_U)\\ &= (R_U^* \otimes \iota_U)(\iota_{\bar U} \otimes \sigma_{U,U})(R_U \otimes \iota_U) = \theta_U.\end{aligned}$$

Now for every object U put $\tilde\theta_U = (\iota \otimes \mathrm{Tr})(\sigma^{-1}_{U,U})$. Then just as for θ one shows that $\tilde\theta_U$ is natural in U and $\sigma_{U,\bar U}\bar R_U = (\iota \otimes \tilde\theta_U)R_U$. It follows that

$$R_U = \sigma_{U,\bar U}(\theta_U \otimes \iota)\bar R_U = (\iota \otimes \theta_U)\sigma_{U,\bar U}\bar R_U = (\iota \otimes \theta_U\tilde\theta_U)R_U,$$

and therefore $\theta_U^{-1} = \tilde\theta_U$.

Step 4. We have $\theta_{\bar U} = (\theta_U)^\vee$ *and* $\dim_q^\theta U = d_i(U)$.

Since $\sigma_{U,\bar U}\bar R_U = (\iota \otimes \theta_U^{-1})R_U$ by Step 3, Lemma 2.6.8 implies that

$$\mathrm{Tr}_U(T\theta_U^{-1}) = \mathrm{Tr}_{\bar U}(T^\vee\theta_{\bar U}^{-1})$$

for all $T \in \mathrm{End}(U)$. On the other hand, it is immediate by definition that $\mathrm{Tr}_U(S) = \mathrm{Tr}_{\bar U}(S^\vee)$ for all $S \in \mathrm{End}(U)$. It follows $\mathrm{Tr}_{\bar U}(T^\vee(\theta_U^{-1})^\vee) = \mathrm{Tr}_{\bar U}(T^\vee\theta_{\bar U}^{-1})$ for all T, whence $\theta_{\bar U} = \theta_U^\vee$.

Since $\sigma_{U,\bar U}(\theta_U \otimes \iota)\bar R_U = R_U$, we have

$$\dim_q^\theta U = R_U^*\sigma_{U,\bar U}(\theta_U \otimes \iota)\bar R_U = \|R_U\|^2 = d_i(U).$$

Step 5. We have $\theta_U = (\iota\otimes \mathrm{Tr}_U)(\sigma_{U,U})$ *and* $\theta_U^{-1} = (\mathrm{Tr}_U\otimes\iota)(\sigma_{U,U}^{-1})$.

Put $\theta'_U = (\iota\otimes \mathrm{Tr}_U)(\sigma_{U,U})$. Thus θ' is the inverse of the twist defined by the braiding σ^{-1}. From Step 3 we know that

$$(\iota\otimes\theta'_U)R_U = \sigma_{\bar U,U}^{-1}\bar R_U = (\theta_{\bar U}\otimes\iota)R_U.$$

Since $\theta=\theta^\vee$, it follows that $\theta'=\theta$. Replacing σ by σ^{-1} we also get

$$(\iota\otimes \mathrm{Tr}_U)(\sigma_{U,U}^{-1}) = (\mathrm{Tr}_U\otimes\iota)(\sigma_{U,U}^{-1}).$$

End of proof. It remains to show that if σ is self-adjoint, then θ is self-adjoint, and if σ is unitary, then θ is unitary. The first statement is immediate, since $(\mathrm{Tr}_U\otimes\iota)(S)^* = (\mathrm{Tr}_U\otimes\iota)(S^*)$. Assume now that σ is unitary. Then

$$\theta_U^{-1} = (\mathrm{Tr}_U\otimes\iota)(\sigma_{U,U}^{-1}) = (\mathrm{Tr}_U\otimes\iota)(\sigma_{U,U})^* = \theta_U^*.$$

Thus θ_U is unitary. □

Let us see what the above result means for a braided category $\mathrm{Rep}\, G$, with braiding defined by an R-matrix $\mathscr{R}$. A collection of natural automorphisms θ_U of U defines an invertible element θ in the center of $\mathscr{U}(G)$ such that $\theta_U = \pi_U(\theta^{-1})$. Then the conditions imposed on θ in order for it to define a twist are

$$\hat\Delta(\theta^{-1}) = (\theta^{-1}\otimes\theta^{-1})\mathscr{R}_{21}\mathscr{R} \ \text{ and }\ \hat S(\theta)=\theta,$$

where the second condition follows from Example 2.2.23. A central element θ with these properties is called a ribbon element. We therefore have a one-to-one correspondence between twists on $(\mathrm{Rep}\, G, \Sigma\mathscr{R})$ and ribbon elements.

By Theorem 2.6.9 there is a canonical ribbon element $\theta\in\mathscr{U}(G)$, given by

$$\pi_U(\theta) = (\iota\otimes \mathrm{Tr}_U)(\sigma_{U,U}^{-1}),$$

where Tr_U is the categorical trace on $\mathrm{End}(U)$, not the canonical trace on $B(H_U)$. In order to compute this element θ it will be convenient to use the following convention similar to Sweedler's sumless notation discussed after the formulation of Lemma 2.3.4. We will work with elements $\omega\in\mathscr{U}(G\times G)$ as if they were finite sums of elementary tensors. Furthermore, we will omit sums, therefore writing simply $\omega=\omega_1\otimes\omega_2$. Recall from Example 2.2.3 that the following defines a standard solution of the conjugate equations for U:

$$R_U1 = \sum_i \bar e_i\otimes\rho^{-1/2}e_i,\ \ \bar R_U 1 = \sum_i \rho^{1/2}e_i\otimes\bar e_i,$$

where we have used module notation and have omitted π_U. We have $\bar{R}_U^*(\xi\otimes\bar{\zeta}) = (\rho^{1/2}\xi,\zeta)$. Hence for any $\xi\in H_U$ we get

$$\begin{aligned}\theta\xi &= (\iota\otimes\bar{R}_U^*)(\mathscr{R}^{-1}\Sigma\otimes\iota)(\iota\otimes\bar{R}_U)\xi\\ &= \sum_i(\iota\otimes\bar{R}_U^*)((\mathscr{R}^{-1})_1\rho^{1/2}e_i\otimes(\mathscr{R}^{-1})_2\xi\otimes\bar{e}_i)\\ &= \sum_i(\rho^{1/2}(\mathscr{R}^{-1})_2\xi,e_i)(\mathscr{R}^{-1})_1\rho^{1/2}e_i\\ &= (\mathscr{R}^{-1})_1\rho(\mathscr{R}^{-1})_2\xi,\end{aligned}$$

so $\theta = (\mathscr{R}^{-1})_1\rho(\mathscr{R}^{-1})_2 = (\mathscr{R}^{-1})_1\hat{S}^2((\mathscr{R}^{-1})_2)\rho$. Since $(\iota\otimes\hat{S})(\mathscr{R}^{-1}) = \mathscr{R}$ by Theorem 2.6.6(iii), we finally get

$$\theta = \mathscr{R}_1\hat{S}(\mathscr{R}_2)\rho.$$

Since $\hat{S}(\theta)=\theta$, the element θ is central and $(\hat{S}\otimes\hat{S})(\mathscr{R}) = \mathscr{R}$, we can also write

$$\theta = \hat{S}(\rho)\hat{S}^2(\mathscr{R}_2)\hat{S}(\mathscr{R}_1) = \rho^{-1}\hat{S}(\mathscr{R}_2)\mathscr{R}_1 = \hat{S}(\mathscr{R}_2)\mathscr{R}_1\rho^{-1}.$$

The element $u = \hat{S}(\mathscr{R}_2)\mathscr{R}_1 = m(\hat{S}\otimes\iota)(\mathscr{R}_{21})$ is often called the Drinfeld element. Note that since θ^{-1} is the twist corresponding to the braiding $(\Sigma\mathscr{R})^{-1} = \Sigma\mathscr{R}_{21}^{-1}$, we have

$$\theta^{-1} = (\mathscr{R}^{-1})_2\hat{S}((\mathscr{R}^{-1})_1)\rho = \hat{S}((\mathscr{R}^{-1})_1)(\mathscr{R}^{-1})_2\rho^{-1}. \tag{2.6.5}$$

Theorem 2.6.9 can now be reformulated as follows.

Theorem 2.6.10. — *Assume $\mathscr{R}$ is an R-matrix for a compact quantum group G, and put $u = m(\hat{S}\otimes\iota)(\mathscr{R}_{21}) \in \mathscr{U}(G)$. Then $\theta = u\rho^{-1}$ is a ribbon element, and $\dim_q^\theta = \dim_q$. If in addition $\mathscr{R}^* = \mathscr{R}_{21}$, then $\theta=\theta^*$, and if $\mathscr{R}$ is unitary, then θ is unitary.*

Example 2.6.11. — Assume Γ is a discrete abelian group, $G=\hat{\Gamma}$. As we discussed in Example 2.6.3, an R-matrix for G is a bi-quasi-character $\mathscr{R}\colon\Gamma\times\Gamma\to\mathbb{C}^*$. Then $\theta(\gamma) = \mathscr{R}(\gamma,\gamma)^{-1}$ for $\gamma\in\Gamma$. □

Example 2.6.12. — Let G_q be the q-deformation of a simply connected semisimple compact Lie group, and $\mathscr{R}=\mathscr{R}_q$ be the R-matrix defined in Example 2.6.4. Let us compute the ribbon element defined in the previous theorem. In this case by Proposition 2.4.10 the Woronowicz character f_1 equals $K_{-2\rho}$. Thus $\theta = \hat{S}_q(\mathscr{R}_2)\mathscr{R}_1K_{2\rho} = \mathscr{R}_1\hat{S}_q(\mathscr{R}_2)K_{-2\rho}$. In order to compute how θ acts on a simple highest weight module V_λ, it suffice to check how it acts on the highest weight vector ξ_λ. We have $K_{-2\rho}\xi_\lambda = q^{-(\lambda,2\rho)}\xi_\lambda$. By virtue of Equation (2.6.2) for $\mathscr{R}_q$ the action of $\mathscr{R}_1\hat{S}_q(\mathscr{R}_2)$ on ξ_λ is the same as if $\mathscr{R}_q$ were equal to $q^{\sum_{i,j}(B^{-1})_{ij}H_i\otimes H_j}$. Since the elements H_i mutually commute and $\hat{S}_q(H_i) = -H_i$, we thus get

$$\mathscr{R}_1\hat{S}_q(\mathscr{R}_2)\xi_\lambda = q^{-\sum_{i,j}(B^{-1})_{ij}H_iH_j}\xi_\lambda.$$

We have

$$\sum_{i,j}(B^{-1})_{ij}H_iH_j\xi_\lambda = \sum_{i,j}(B^{-1})_{ij}(\lambda,\alpha_i^\vee)(\lambda,\alpha_j^\vee)\xi_\lambda = (\lambda,\lambda)\xi_\lambda.$$

Therefore, putting everything together, we get

$$\pi_\lambda(\theta) = \pi_\lambda(\mathscr{R}_1\hat{S}_q(\mathscr{R}_2)K_{-2\rho}) = q^{-(\lambda,\lambda+2\rho)}1.$$

Denote by $C_q \in \mathscr{U}(G_q)$ the central element that acts on every simple highest weight module V_λ as the scalar $(\lambda,\lambda+2\rho)$. Recall that in the classical case this is exactly how the Casimir operator acts. We have thus proved that $\theta = q^{-C_q}$, and therefore

$$\mathscr{R}_q^*\mathscr{R}_q = q^{\hat{\Delta}_q(C_q)}(q^{-C_q}\otimes q^{-C_q}). \tag{2.6.6}$$ □

The ribbon element defined in Theorem 2.6.10 is in general not the only one possible. Indeed, if $z \in \mathscr{U}(G)$ is a central group-like element such that $\hat{S}(z) = z$, then $zu\rho^{-1}$ again defines a twist on $(\operatorname{Rep} G, \Sigma\mathscr{R})$, and this way we get all possible ribbon elements. Note that if z is group-like, then $\hat{S}(z) = z^{-1}$, so the condition $\hat{S}(z) = z$ is equivalent to $z^2 = 1$. As a consequence, for any choice of a twist on $\operatorname{Rep} G$ we have

$$\dim_q^\theta U = \pm \dim_q U$$

for any irreducible representation U. We also conclude that the ribbon element θ defined by Theorem 2.6.10 is the unique ribbon element such that the quantum dimension $\dim_q^\theta$ is positive-valued.

It is curious that zu is itself the Drinfeld element for a modified R-matrix. Namely, consider the element

$$\mathscr{R}_z = (1 + 1\otimes z + z\otimes 1 - z\otimes z)/2.$$

It is easy to check that upon identifying $\{1, z\}$ with $\mathbb{Z}/2\mathbb{Z}$ this is the unique nontrivial R-matrix for $\mathbb{Z}/2\mathbb{Z}$ such that $\mathscr{R}_z^* = (\mathscr{R}_z)_{21}$. Then $\tilde{\mathscr{R}} = \mathscr{R}_z\mathscr{R}$ is again an R-matrix for G, and if $\mathscr{R}^* = \mathscr{R}_{21}$, then we still have $\tilde{\mathscr{R}}^* = \tilde{\mathscr{R}}_{21}$. The corresponding element $\tilde{u} = \hat{S}(\tilde{\mathscr{R}}_2)\tilde{\mathscr{R}}_1$ equals zu.

If $\mathscr{R}^* = \mathscr{R}_{21}$, then in addition to $\theta = u\rho^{-1}$ another natural choice for a ribbon element is $|\theta|$. The fact that $|\theta|$ is indeed a ribbon element follows by applying the polar decomposition to the identity

$$\mathscr{R}^*\mathscr{R} = (\theta\otimes\theta)\hat{\Delta}(\theta)^{-1}.$$

An advantage of $|\theta|$ is that it is the unique element defining a positive twist on $(\operatorname{Rep} G, \Sigma\mathscr{R})$. For the q-deformation G_q of a compact Lie group G the element $\theta = u\rho^{-1}$ happens to be positive, but clearly this is not true in general.

References. — [**18**], [**48**], [**59**], [**100**].

2.7. AMENABILITY

As we discussed in Section 1.7, any compact quantum group has a reduced form $(C_r(G), \Delta_r)$ and a universal form $(C_u(G), \Delta_u)$. In this section we will find necessary and sufficient conditions for these two forms to coincide. If G is the dual of a discrete group Γ, then $C_r(G) = C_r^*(\Gamma)$ and $C_u(G) = C^*(\Gamma)$. In this case it is well-known that $C^*(\Gamma) = C_r^*(\Gamma)$ if and only if Γ is amenable. We will extend this result to quantum groups by developing a notion of amenability for discrete quantum groups. Remarkably, amenability can be detected simply by looking at fusion rules for G, that is, at how the tensor product of two representations decomposes into irreducible representations. We will therefore start with more general categorical considerations.

Throughout the whole section we will assume that $\mathscr{C}$ is a strict C*-tensor category with conjugates.

Definition 2.7.1. — A dimension function on $\mathscr{C}$ is a map d that assigns a nonnegative number $d(U)$ to every object U in $\mathscr{C}$ such that $d(U) > 0$ if U is nonzero, $d(U) = d(V)$ if $U \cong V$,

$$d(U \oplus V) = d(U) + d(V), \;\; d(U \otimes V) = d(U)d(V), \;\; \text{and} \;\; d(\bar{U}) = d(U).$$

Note that since $\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$ and $\mathbb{1}$ is a subobject of $U \otimes \bar{U}$ for every nonzero U, we automatically have $d(\mathbb{1}) = 1$ and $d(U) \geq 1$.

Clearly, the intrinsic dimension d_i is an example of a dimension function.

The notion of a dimension function depends on a much rougher structure than C*-tensor categories.

Definition 2.7.2. — The fusion ring of $\mathscr{C}$, which we denote by $K(\mathscr{C})$, is the universal ring generated by elements $[U]$, $U \in \operatorname{Ob}\mathscr{C}$, such that $[U] = [V]$ if $U \cong V$, $[U] + [V] = [U \oplus V]$ and $[U][V] = [U \otimes V]$.

Fix representatives U_j, $j \in I$, of isomorphism classes of simple objects. Denote by $e \in I$ the point corresponding to $\mathbb{1}$. Then we can identify $K(\mathscr{C})$ with $\oplus_{j\in I}\mathbb{Z}j$, with multiplication given by

$$ij = \sum_k m_{ij}^k k, \;\; \text{if} \;\; U_i \otimes U_j \cong \bigoplus_k m_{ij}^k U_k.$$

Define an involution $j \mapsto \bar{j}$ on I such that $U_{\bar{j}} \cong \bar{U}_j$. Then we can say that a dimension function is a homomorphism $d\colon K(\mathscr{C}) \to \mathbb{R}$ such that

$$d(j) > 0 \;\; \text{and} \;\; d(\bar{j}) = d(j) \;\; \text{for all} \;\; j \in I.$$

For every $j \in I$ denote by Λ_j the operator of multiplication by j on the left in the algebra $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathscr{C}) = \oplus_{k\in I}\mathbb{C}k$. Our goal is to show that Λ_j extends to a bounded operator on $\ell^2(I)$. The proof is based on the following lemma.

Lemma 2.7.3. — *Let $\Gamma = (\gamma_{ik})_{i,k\in I}$ be a matrix with nonnegative real coefficients. Assume there exists a vector $u = (u_i)_{i\in I}$ such that $u_i > 0$ for all $i \in I$, the vector $v = \Gamma u$ is well-defined and $\Gamma^t v \leq u$ coordinate-wise. Then Γ defines a contraction on $\ell^2(I)$.*

Proof. — Note that if $v_i = 0$ for some i, then $\gamma_{ik} = 0$ for all k. Thus replacing $v_i = 0$ by any strictly positive number for every such i we get a vector v such that $v_i > 0$ for all i, $\Gamma u \leq v$ and $\Gamma^t v \leq u$. Then for vectors $\xi = (\xi_i)_i$ and $\zeta = (\zeta_i)_i$ in $\ell^2(I)$, by the Cauchy-Schwarz inequality we get

$$\begin{aligned}|(\Gamma\xi,\zeta)| = \left|\sum_{i,k}\gamma_{ik}\xi_k\bar{\zeta}_i\right| &= \left|\sum_{i,k}\left((\gamma_{ik}v_iu_k^{-1})^{1/2}\xi_k\right)\left((\gamma_{ik}u_kv_i^{-1})^{1/2}\bar{\zeta}_i\right)\right|\\ &\leq \left(\sum_{i,k}\gamma_{ik}v_iu_k^{-1}|\xi_k|^2\right)^{1/2}\left(\sum_{i,k}\gamma_{ik}u_kv_i^{-1}|\zeta_i|^2\right)^{1/2} \leq \|\xi\|\|\zeta\|,\end{aligned}$$

so that $\|\Gamma\| \leq 1$. □

Proposition 2.7.4. — *The operator Λ_j extends to a bounded linear operator on $\ell^2(I)$. Furthermore, for every dimension function d on $\mathscr{C}$ we have $\|\Lambda_j\| \leq d(j)$.*

Proof. — We apply the previous lemma to the matrix $\Gamma = (m^i_{jk}/d(j))_{i,k}$ defining the operator $\Lambda_j/d(j)$, and $u_i = v_i = d(i)$. Then $\Gamma^t v = u$ by definition of a dimension function. Since $\gamma_{ki} = m^k_{ji}/d(j) = m^i_{\bar{j}k}/d(j)$ by the Frobenius reciprocity, we also have $\Gamma u = v$. □

Fix now a dimension function d on $\mathscr{C}$. For a probability measure μ on I define a contraction λ_μ on $\ell^2(I)$ by

$$\lambda_\mu = \sum_{j\in I}\frac{\mu(j)}{d(j)}\Lambda_j.$$

We will write λ_j instead of λ_{δ_j}. If μ and ν are two probability measures, then $\lambda_\mu\lambda_\nu = \lambda_{\mu*\nu}$, where

$$(\mu*\nu)(k) = \sum_{i,j} m^k_{ij}\frac{d(k)}{d(i)d(j)}\mu(i)\nu(j).$$

We will write μ^n for the n-th convolution power $\mu*\cdots*\mu$ of μ. For a measure μ denote by $\check{\mu}$ the measure defined by $\check{\mu}(i) = \mu(\bar{i})$. Then $\lambda_\mu^* = \lambda_{\check{\mu}}$. A probability measure μ on I is called symmetric if $\check{\mu} = \mu$, and it is called nondegenerate if $\cup_{n\geq 1}\operatorname{supp}\mu^n = I$.

Lemma 2.7.5. — *For a probability measure μ on I consider the following conditions:*

(i) $1 \in \operatorname{Sp}\lambda_\mu$*;*

(ii) $\|\lambda_\mu\| = 1$*;*

(iii) $(\check{\mu}*\mu)^n(e)^{1/n} \to 1$ *as* $n \to +\infty$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) *. If* μ *is symmetric, then all three conditions are equivalent, and if they are satisfied, there exists a sequence* $\{\xi_n\}_n$ *of unit vectors in* $\ell^2(I)$ *such that* $\|\lambda_j\xi_n-\xi_n\|\to 0$ *as* $n\to+\infty$ *for all* $j\in\cup_{n\geq1}\operatorname{supp}\mu^n$*; in particular,* $1\in\operatorname{Sp}\lambda_\nu$ *for any probability measure* ν *such that* $\operatorname{supp}\nu$ *is contained in* $\cup_{n\geq1}\operatorname{supp}\mu^n$.

Proof. — It is clear that (i)$\Rightarrow$(ii), and since $(\check\mu*\mu)^n(e)=((\lambda_\mu^*\lambda_\mu)^n\delta_e,\delta_e)$, that (iii)$\Rightarrow$(ii).

In order to show that (ii)$\Rightarrow$(iii), consider the unital C*-algebra A generated by $\lambda_\mu^*\lambda_\mu$. Similarly to the proof of Proposition 2.7.4, by Lemma 2.7.3 the operators of multiplication by $j\in I$ on the right on $K(\mathscr{C})$ extend to bounded operators on $\ell^2(I)$. It follows that the vector δ_e is cyclic for the commutant A' of A, hence $(\cdot\,\delta_e,\delta_e)$ is a faithful state on A. From this we conclude that $\|\lambda_\mu\|^2$ is the least upper bound of the support of the measure ν on $\operatorname{Sp}\lambda_\mu^*\lambda_\mu$ defined by the state $(\cdot\,\delta_e,\delta_e)$. Since

$$((\lambda_\mu^*\lambda_\mu)^n\delta_e,\delta_e)=\int t^n d\nu(t)\ \text{ for all }\ n\geq 0,$$

it is easy to see that this upper bound is equal to

$$\lim_{n\to+\infty}((\lambda_\mu^*\lambda_\mu)^n\delta_e,\delta_e)^{1/n}=\lim_{n\to+\infty}(\check\mu*\mu)^n(e)^{1/n}.$$

Hence (ii)$\Rightarrow$(iii).

Next assume that μ is symmetric and condition (ii) is satisfied. Since λ_μ is self-adjoint and $\|\lambda_\mu\|=1$, there exists a sequence of unit vectors $\zeta_n\in\ell^2(I)$ such that $|(\lambda_\mu\zeta_n,\zeta_n)|\to 1$. Consider the unit vectors $\xi_n=|\zeta_n|$. Since the matrix λ_μ has non-negative coefficients, we have $(\lambda_\mu\xi_n,\xi_n)\geq|(\lambda_\mu\zeta_n,\zeta_n)|$. Hence $(\lambda_\mu\xi_n,\xi_n)\to 1$, and therefore $1\in\operatorname{Sp}\lambda_\mu$.

Since λ_μ is a convex combination of the operators λ_j, we also see that

$$(\lambda_j\xi_n,\xi_n)\to 1$$

for every $j\in\operatorname{supp}\mu$. Since λ_j is a contraction, this implies that $\|\lambda_j\xi_n-\xi_n\|\to 0$. Since $\|\lambda_\mu^k\xi_n-\xi_n\|\to 0$ for all $k\geq 1$, we similarly conclude that $\|\lambda_j\xi_n-\xi_n\|\to 0$ for every $j\in\operatorname{supp}\mu^k$. ☐

The previous lemma allows us to introduce the following notion.

Definition 2.7.6. — The pair $(K(\mathscr{C}),d)$ is called amenable if the following equivalent conditions are satisfied:

(i) $1\in\operatorname{Sp}\lambda_\mu$ for every probability measure μ;

(ii) $\|\lambda_\mu\|=1$ for every probability measure μ;

(iii) $(\check\mu*\mu)^n(e)^{1/n}\to 1$ as $n\to+\infty$ for every probability measure μ;

(iv) there exists a net $\{\xi_\alpha\}_\alpha$ of unit vectors in $\ell^2(I)$ such that

$$\|\lambda_j\xi_\alpha - \xi_\alpha\| \underset{\alpha}{\to} 0 \text{ for all } j.$$

The category $\mathscr{C}$ is called amenable if $(K(\mathscr{C}), d_i)$ is amenable, where d_i is the intrinsic dimension function.

Note that Lemma 2.7.5 implies that it suffices to check any of conditions (i)-(iii) in the above definition for a net $\{\mu_\alpha\}_\alpha$ of symmetric probability measures such that the sets $\cup_{n\geq 1}\operatorname{supp}\mu_\alpha^n$ increase with α and their union is I. In particular, if I is countable, it suffices to consider one nondegenerate symmetric probability measure.

Proposition 2.7.7. — *Let d be a dimension function on a C^*-tensor category $\mathscr{C}$ such that $(K(\mathscr{C}), d)$ is amenable. Then $d(j) = \|\Lambda_j\| = \|(m^i_{jk})_{i,k}\|$ for every $j \in I$, and for any other dimension function d' on $\mathscr{C}$ we have $d' \geq d$.*

Proof. — For $(K(\mathscr{C}), d)$ to be amenable, at the very least we need the condition $\|\lambda_j\| = 1$ to be satisfied for every $j \in I$, but it means exactly that $d(j) = \|\Lambda_j\|$. The second statement follows from the inequality $\|\Lambda_j\| \leq d'(j)$, which by Proposition 2.7.4 holds for any dimension function d'. □

Corollary 2.7.8. — *If $\mathscr{C}$ is finite, meaning that the set I of isomorphism classes of simple objects is finite, then $\mathscr{C}$ is amenable and the intrinsic dimension is the only dimension function on $\mathscr{C}$.*

Proof. — Any dimension function d on $\mathscr{C}$ is amenable, since the vector $(d(j))_{j\in I}$ is an eigenvector of λ_μ with eigenvalue 1 for every probability measure μ. Hence $d(j) = \|\Lambda_j\|$. □

Corollary 2.7.9. — *Let $\mathscr{C}$ and $\mathscr{C}'$ be C^*-tensor categories with conjugates and $F\colon \mathscr{C} \to \mathscr{C}'$ be a unitary tensor functor. Assume $\mathscr{C}$ is amenable. Then $d_i(F(U)) = d_i(U)$ for every object U in $\mathscr{C}$.*

Proof. — Consider the dimension function $d = d_iF$ on $\mathscr{C}$. By Corollary 2.2.20 we have $d \leq d_i$. On the other hand, d_i is the smallest dimension function on $\mathscr{C}$ by amenability. Hence $d = d_i$. □

We now turn to the promised application of amenability to quantum groups. For a compact quantum group G, denote by h the Haar state on $C_u(G)$. By definition the image of the GNS-representation π_h of $C_u(G)$ is $C_r(G)$.

For every finite dimensional unitary representation U of G put

$$\chi(U) = (\mathrm{Tr} \otimes \iota)(U) \in \mathbb{C}[G].$$

This element is called the character of U. Note that the linear span of characters is a unital $*$-subalgebra of $\mathbb{C}[G]$: $\chi(U)^* = \chi(\bar{U})$ and $\chi(U)\chi(V) = \chi(U \times V)$.

Theorem 2.7.10. — *For a compact quantum group G the following conditions are equivalent:*

(i) *the fusion ring of G, that is, of* $\operatorname{Rep} G$, *together with the classical dimension function* dim *is amenable;*

(ii) *for every finite dimensional unitary representation U of G we have* $\dim U \in \operatorname{Sp} \pi_h(\chi(U))$*;*

(iii) *the counit $\varepsilon\colon \mathbb{C}[G] \to \mathbb{C}$ extends to a bounded linear functional on $C_r(G)$;*

(iv) *the map $\pi_h\colon C_u(G) \to C_r(G)$ is an isomorphism.*

Proof. — (i)$\Leftrightarrow$(ii) Consider the norm closure A of the linear span of the operators $\pi_h(\chi(U))$ in $C_r(G)$. This is a unital C*-algebra. Put $H = \overline{A\xi_h} \subset L^2(G)$. Since by Corollary 1.7.5 the Haar state is faithful on $C_r(G)$, we have $\dim U \in \operatorname{Sp} \pi_h(\chi(U))$ if and only if $\dim U \in \operatorname{Sp}(\pi_h(\chi(U))|_H)$.

Choose representatives U_j, $j \in I$, of isomorphism classes of irreducible unitary representations of G. By the orthogonality relations the map $T\colon \ell^2(I) \to H$, $T\delta_j = \chi(U_j)\xi_h$, is a unitary isomorphism. Furthermore, if $U \cong \oplus_j n_j U_j$ and $\mu = (\dim U)^{-1} \sum_j n_j (\dim U_j)\delta_j$, then one can easily check that $\chi(U)T = (\dim U)T\lambda_\mu$. Therefore $1 \in \operatorname{Sp}\lambda_\mu$ if and only if $\dim U \in \operatorname{Sp}(\pi_h(\chi(U))|_H)$. Note also that if U is self-conjugate, then μ is symmetric. Since the supports of such measures μ form a directed set with union equal to I, we conclude that (i) and (ii) are equivalent.

(ii)$\Leftrightarrow$(iii) Assume (iii) holds. Then $\varepsilon(\chi(U)) = \dim U$ as $(\iota \otimes \varepsilon)(U) = 1$, and since ε is a character of the C*-algebra $C_r(G)$, it follows that $\dim U \in \operatorname{Sp} \pi_h(\chi(U))$.

Assume (ii) holds. Let U be a self-conjugate unitary representation. Since $\dim U \in \operatorname{Sp} \pi_h(\chi(U))$ and $\chi(U)$ is self-adjoint, there exists a state ω on $C_r(G)$ such that $\omega(\chi(U)) = \dim U$. Since the inequality $\|\chi(V)\| \le \dim V$ holds for any representation V, we conclude that $\omega(\chi(V)) = \dim V$ for any subrepresentation V of U. Choosing an increasing net of self-conjugate unitary representations and taking a weak* limit point of the corresponding states ω, we get a state ν on $C_r(G)$ such that $\nu(\chi(V)) = \dim V$ for any V.

Next, for every finite dimensional unitary representation V, the operator $X = (\iota \otimes \nu)(V) \in B(H_V)$ has the properties $\|X\| \le 1$ and $\operatorname{Tr} X = \dim H_V$, which is possible only when $X = 1$. Hence ν is an extension of the counit ε on $\mathbb{C}[G]$.

(iii)$\Leftrightarrow$(iv) The implication (iv)$\Rightarrow$(iii) is obvious, as the counit is well-defined on $C_u(G)$.

In order to prove that (iii)$\Rightarrow$(iv), observe that the comultiplication $\Delta\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ extends to a $*$-homomorphism $\alpha\colon C_r(G) \to C_r(G) \otimes C_u(G)$. Indeed, consider the right regular representation $W \in M(K(L^2(G)) \otimes C_u(G))$. Then we can define $\alpha(a) = W(a \otimes 1)W^*$ for $a \in C_r(G)$, see Theorem 1.5.3(ii).

Then $\vartheta = (\varepsilon\otimes\iota)\alpha\colon C_r(G)\to C_u(G)$ is a unital $*$-homomorphism such that $\vartheta(\pi_h(a))=a$ for $a\in\mathbb{C}[G]$. Hence ϑ is the inverse of $\pi_h\colon C_u(G)\to C_r(G)$. □

Note that by the proof of the theorem and by Lemma 2.7.5 condition (ii) can be replaced by several other equivalent conditions. For example, if there exists a self-conjugate finite dimensional unitary representation U such that the matrix coefficients of U generate $\mathbb{C}[G]$, then we may require $\dim U \in \operatorname{Sp}\pi_h(\chi(U))$ or $\|\pi_h(\chi(U))\| = \dim U$. We refer the reader to [**11**], [**40**] and [**78**] for some other equivalent conditions.

Definition 2.7.11. — A compact quantum group G is called coamenable if the equivalent conditions in Theorem 2.7.10 are satisfied.

We want to stress that coamenability of G is in general not the same as amenability of the category $\operatorname{Rep} G$ of finite dimensional representations of G, since coamenability refers to the classical dimension and not the quantum dimension. The two notions of dimension coincide if and only if G is of Kac type.

Proposition 2.7.12. — *Any compact group G is coamenable.*

Proof. — Clearly, $C_r(G)=C(G)$ and the counit $\varepsilon\colon C(G)\to\mathbb{C}$ is well-defined: it is the evaluation at the unit element of G. □

Therefore for any compact group G the category $\operatorname{Rep} G$ is amenable. By virtue of Corollary 2.7.9 we then get the following result.

Corollary 2.7.13. — *If G is a compact group and $F\colon \operatorname{Rep} G\to \mathrm{Hilb}_f$ is a unitary fiber functor, then $\dim F(U)=\dim U$ for any finite dimensional unitary representation U.*

Note that as follows from Corollary 2.5.4, the above corollary is not true for quantum groups, specifically, for free orthogonal quantum groups.

Let us now consider some examples.

Theorem 2.7.14. — *The q-deformation G_q, $q>0$, of any simply connected semisimple compact Lie group G is coamenable.*

Proof. — For $q=1$ the result is true by Proposition 2.7.12. On the other hand, by Theorem 2.4.7 the fusion ring of G_q together with the classical dimension function does not depend on q. □

Therefore $\mathbb{C}[G_q]$ has only one completion to a C*-algebra of continuous functions on a compact quantum group, a result that we promised in Section 2.4.

Theorem 2.7.15. — *The free orthogonal quantum group $A_o(F)$ defined by a matrix $F\in \mathrm{GL}_n(\mathbb{C})$ (such that $F\bar F=\pm1$) is coamenable for $n=2$ and non-coamenable for $n\geq 3$.*

Proof. — By Theorem 2.5.11 the compact quantum group $A_o(F)$ is monoidally equivalent to $SU_q(2)$ for some $q \in [-1,1]\setminus\{0\}$, and under this equivalence the fundamental representation of $A_o(F)$ corresponds to the fundamental representation of $SU_q(2)$. In other words, the fusion ring of $A_o(F)$ is that of $SU(2)$, and the dimension of the fundamental representation is n. If $n = 2$, we therefore get the classical dimension function for $SU(2)$, hence $A_o(F)$ is coamenable (and, as follows from Theorem 2.5.11 and Remark 2.5.5, isomorphic to $SU_q(2)$ for some q). If $n \geq 3$, we get a strictly larger dimension function, hence $A_o(F)$ is not coamenable. □

It can also be shown that the free unitary groups $A_u(F)$ are always non-coamenable [**4**], while the free permutation groups $A_s(n)$ are coamenable for $n \leq 4$ and non-coamenable for $n \geq 5$ [**7**].

References. — [**4**], [**6**], [**5**], [**7**], [**10**], [**11**], [**34**], [**40**], [**56**], [**59**], [**69**], [**78**].

CHAPTER 3

COHOMOLOGY OF QUANTUM GROUPS

In this chapter we develop a low dimensional cohomology theory for discrete quantum groups. The principal result is a computation of some of these groups for the q-deformations of semisimple Lie groups. The result itself, as well as the technique used to prove it, will play a crucial role in the next chapter.

3.1. DUAL COCYCLES

Assume Γ is a discrete group. Then the cohomology groups $H^n(\Gamma;\mathbb{C}^*)$ of Γ with coefficients in $\mathbb{C}^*$ can be computed using the cobar complex

$$1 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \dots,$$

where C^n is the group of functions $\Gamma^n \to \mathbb{C}^*$ with pointwise multiplication and the differential $\partial\colon C^n \to C^{n+1}$ is defined by

$$\partial f = \prod_{k=0}^{n+1} (\partial_n^k f)^{(-1)^k},$$

where

$$\begin{aligned}
(\partial_n^0 f)(\gamma_1,\dots,\gamma_{n+1}) &= f(\gamma_2,\dots,\gamma_{n+1}),\\
(\partial_n^{n+1} f)(\gamma_1,\dots,\gamma_{n+1}) &= f(\gamma_1,\dots,\gamma_n),\\
(\partial_n^k f)(\gamma_1,\dots,\gamma_{n+1}) &= f(\gamma_1,\dots,\gamma_{k-1},\gamma_k\gamma_{k+1},\gamma_{k+2},\dots,\gamma_n) \ \text{ for } \ 1\le k\le n.
\end{aligned}$$

Let us now try to write a similar complex for discrete quantum groups. Assume that G is a compact quantum group. For $n\ge 0$ and an invertible element $\omega\in\mathscr{U}(G^n)$ (we let $\mathscr{U}(G^0)=\mathbb{C}$) define

$$\partial_n^0\omega = 1\otimes\omega,\ \ \partial_n^{n+1}\omega=\omega\otimes 1,$$

$$\partial_n^k\omega = (\iota\otimes\cdots\otimes\iota\otimes\hat\Delta\otimes\iota\otimes\cdots\otimes\iota)(\omega)\ \text{ for } \ 1\le k\le n,$$

with $\hat{\Delta}$ in the k-th position. Then put

$$\partial\omega = \partial_n^0(\omega)\partial_n^2(\omega)\dots\partial_n^1(\omega^{-1})\partial_n^3(\omega^{-1})\dots.$$

For general compact quantum groups this way we no longer get a complex. Nevertheless for small n we still get something meaningful. Before we turn to this, let us introduce more notation.

For invertible elements $\chi\in\mathscr{U}(G^{n+1})$ and $\omega\in\mathscr{U}(G^n)$ put

$$\chi_\omega = (\partial_n^0(\omega)\partial_n^2(\omega)\dots)\chi(\partial_n^1(\omega^{-1})\partial_n^3(\omega^{-1})\dots),$$

so that $1_\omega = \partial\omega$.

Definition 3.1.1. — A dual $\mathbb{C}^*$-valued (resp. $\mathbb{T}$-valued) n-cocycle on G, or a $\mathbb{C}^*$-valued (resp. $\mathbb{T}$-valued) n-cocycle on $\hat{G}$, is an invertible (resp. unitary) element χ of $\mathscr{U}(G^n)$ such that $\partial\chi = 1$.

A cocycle χ is called normalized, or counital, if applying $\hat{\varepsilon}$ to any of the factors of χ we get 1.

Given two invertible (resp. unitary) elements χ' and χ of $\mathscr{U}(G^n)$ we say that χ' is cohomologous to χ if there exists an invertible (resp. unitary) element $\omega\in\mathscr{U}(G^{n-1})$ such that $\chi' = \chi_\omega$. Elements of the form $\partial\omega$ are called coboundaries.

We stress again that for general quantum groups these notions do not have properties one would like to have: coboundaries are not necessarily cocycles, the relation of being cohomologous in not symmetric, and so on.

Let us now introduce a class of cocycles that we will be particularly interested in.

Definition 3.1.2. — An element of $\mathscr{U}(G^n)$ is said to be invariant if it commutes with the elements in the image of $\hat{\Delta}^{(n-1)}\colon\mathscr{U}(G)\to\mathscr{U}(G^n)$, where $\hat{\Delta}^{(n-1)}$ is defined inductively as follows: $\hat{\Delta}^{(1)} = \hat{\Delta}$, and $\hat{\Delta}^{(k)}(\omega)$ is obtained by applying $\hat{\Delta}$ to any of the factors of $\hat{\Delta}^{(k-1)}(\omega)$ (this gives the same element by coassociativity of $\hat{\Delta}$).

Given two invertible (resp. unitary) invariant elements χ' and χ of $\mathscr{U}(G^n)$ we say that χ' is cohomologous to χ if there exists an invertible (resp. unitary) invariant element $\omega\in\mathscr{U}(G^{n-1})$ such that $\chi' = \chi_\omega$.

Let us consider what the above definitions mean for $n = 1, 2, 3$.

1-cocycles

A 1-cocycle on $\hat{G}$ is an invertible element u of $\mathscr{U}(G)$ such that

$$\hat{\Delta}(u) = u\otimes u.$$

Such elements are also called group-like. They form a group, which we denote by $H^1(\hat{G};\mathbb{C}^*)$. Note that instead of invertibility it suffices to assume that $u\neq 0$, since the identity $\hat{\Delta}(u) = u \otimes u$ implies that $u = \hat{\varepsilon}(u)u$ and $\varepsilon(u)1 = u\hat{S}(u)$. Unitary group-like elements form a group $H^1(\hat{G};\mathbb{T})$. This group is also called the intrinsic group of G, or more precisely, the intrinsic group of $(\mathscr{U}(G), \hat{\Delta})$.

Invariant elements of $\mathscr{U}(G)$ are exactly the central elements. Considering central group-like elements we get two more groups, which we denote by $H^1_G(\hat{G};\mathbb{C}^*)$ and $H^1_G(\hat{G};\mathbb{T})$.

These four groups have categorical interpretations. Namely, recall that we can identify $\mathscr{U}(G)$ with the algebra of endomorphisms of the forgetful functor $\operatorname{Rep} G \to \mathrm{Hilb}_f$. Then $H^1(\hat{G};\mathbb{C}^*)$ is the group of monoidal automorphisms of the canonical fiber functor $\operatorname{Rep} G \to \mathrm{Hilb}_f$, while $H^1(\hat{G};\mathbb{T})$ is the group of unitary monoidal automorphisms of this tensor functor. Similarly, the group $H^1_G(\hat{G};\mathbb{C}^*)$ (resp. $H^1_G(\hat{G};\mathbb{T})$) is the group of all (resp. all unitary) monoidal automorphisms of the identity tensor functor on $\operatorname{Rep} G$.

Another way of looking at group-like elements is that they are nonzero homomorphisms $\mathbb{C}[G] \to \mathbb{C}$. If u is group-like, then $\hat{S}(u) = u^{-1}$, and therefore

$$u(a^*) = \overline{\hat{S}(u)^*(a)} = \overline{(u^{-1})^*(a)} \quad \text{for all } a \in \mathbb{C}[G].$$

Hence u is unitary if and only if $u\colon \mathbb{C}[G] \to \mathbb{C}$ is a $*$-homomorphism. It follows that we can also identify $H^1(\hat{G};\mathbb{T})$ with the characters of the C*-algebra $C_u(G)$.

In the next section we will find all group-like elements in many cases.

2-cocycles

A 2-cocycle on $\hat{G}$ is an invertible element $\mathscr{E} \in \mathscr{U}(G \times G)$ such that

$$(\mathscr{E} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathscr{E}) = (1 \otimes \mathscr{E})(\iota \otimes \hat{\Delta})(\mathscr{E}).$$

Invertible elements $\mathscr{E}, \mathscr{F} \in \mathscr{U}(G \times G)$ are cohomologous if there exists an invertible element $u \in \mathscr{U}(G)$ such that

$$\mathscr{E} = \mathscr{F}_u = (u \otimes u)\mathscr{F}\hat{\Delta}(u)^{-1}.$$

Then $\partial\mathscr{E} = (u \otimes u \otimes u)\partial\mathscr{F}(u^{-1} \otimes u^{-1} \otimes u^{-1})$, which shows that if $\mathscr{F}$ is a cocycle, then so is $\mathscr{E}$. In particular, $\partial u = (u \otimes u)\hat{\Delta}(u)^{-1}$ is a 2-cocycle.

Example 3.1.3. — Any R-matrix for G is a dual 2-cocycle on G: the cocycle identity is exactly the Yang-Baxter equation. □

For 2-cocycles the property of being cohomologous is an equivalence relation. Denote by $H^2(\hat{G};\mathbb{C}^*)$ the set of equivalence classes of 2-cocycles. The product of two 2-cocycles is not necessarily a cocycle, so $H^2(\hat{G};\mathbb{C}^*)$ is just a set. Restricting to unitary and/or invariant elements we get three more sets $H^2(\hat{G};\mathbb{T})$, $H^2_G(\hat{G};\mathbb{C}^*)$ and $H^2_G(\hat{G};\mathbb{T})$. Note that invariant 2-cocycles do form a group under multiplication, and coboundaries of invertible central elements in $\mathscr{U}(G)$ form a central subgroup of this group. It follows that $H^2_G(\hat{G};\mathbb{C}^*)$ and $H^2_G(\hat{G};\mathbb{T})$ are groups.

Applying $\iota\otimes\hat{\varepsilon}\otimes\iota$ to $(\mathscr{E}\otimes 1)(\hat{\Delta}\otimes\iota)(\mathscr{E}) = (1\otimes\mathscr{E})(\iota\otimes\hat{\Delta})(\mathscr{E})$ we get
$$(\iota\otimes\hat{\varepsilon})(\mathscr{E})\otimes 1 = 1\otimes(\hat{\varepsilon}\otimes\iota)(\mathscr{E}).$$
It follows that
$$(\iota\otimes\hat{\varepsilon})(\mathscr{E}) = (\hat{\varepsilon}\otimes\iota)(\mathscr{E}) = (\hat{\varepsilon}\otimes\hat{\varepsilon})(\mathscr{E})1.$$
Therefore any 2-cocycle is cohomologous to a counital cocycle. Counital dual 2-cocycles are often called twists, or Drinfeld twists, but we will mean by a Drinfeld twist a different object.

The set $H^2(\hat{G};\mathbb{C}^*)$ again allows for a categorical interpretation. Consider the forgetful functor $F\colon \mathrm{Rep}(G)\to\mathrm{Hilb}_f$. Any invertible element $\mathscr{E}\in\mathscr{U}(G\times G)$ defines a natural isomorphism
$$F_2\colon F(U)\otimes F(V)\xrightarrow{\mathscr{E}^{-1}} F(U\otimes V).$$
Then $\mathscr{E}$ is a cocycle if and only if (F,F_2) is a tensor functor, that is, the diagram
$$\begin{tikzcd}
F(U)\otimes F(V)\otimes F(W) \arrow[r, "\mathscr{E}^{-1}\otimes 1"] \arrow[d, "1\otimes\mathscr{E}^{-1}"'] & F(U\otimes V)\otimes F(W) \arrow[d, "(\hat{\Delta}\otimes\iota)(\mathscr{E}^{-1})"] \\
F(U)\otimes F(V\otimes W) \arrow[r, "(\iota\otimes\hat{\Delta})(\mathscr{E}^{-1})"'] & F(U\otimes V\otimes W)
\end{tikzcd}$$
commutes. Furthermore, two 2-cocycles are cohomologous if and only if their corresponding tensor functors $\mathrm{Rep}\,G\to\mathrm{Hilb}_f$ are naturally monoidally isomorphic.

Note now that any functor $E\colon \mathrm{Rep}\,G\to\mathrm{Hilb}_f$ such that $\dim E(U)=\dim U$ for all U is naturally isomorphic to the forgetful functor. If E is in addition a tensor functor, then the tensor structure on E defines a tensor structure on the forgetful functor F, hence a 2-cocycle. It is easy to check that the cohomology class of this cocycle does not depend on the choice of an isomorphism $E\cong F$.

To summarize, the set $H^2(\hat{G};\mathbb{C}^*)$ can be identified with the set of natural monoidal isomorphism classes of dimension-preserving fiber functors $E\colon \mathrm{Rep}\,G\to\mathrm{Hilb}_f$. Similarly, the set $H^2(\hat{G};\mathbb{T})$ can be identified with the set of natural unitary monoidal isomorphism classes of dimension-preserving unitary fiber functors. Recall in passing that by Corollary 2.7.13, if G is a genuine group, then any unitary fiber functor on $\mathrm{Rep}\,G$ is dimension-preserving.

Another important interpretation of $H^2(\hat{G};\mathbb{T})$, which we are not going to discuss (but see Example 3.1.8), is that this is the set of isomorphism classes of full multiplicity ergodic actions of G on von Neumann algebras [**14, 57, 92**].

Example 3.1.4. — Consider the quantum group $SU_q(2)$, $q\in[-1,1]\setminus\{0\}$. As follows from Corollary 2.5.4 and Remark 2.5.5, the canonical fiber functor
$$\mathrm{Rep}\,SU_q(2)\to\mathrm{Hilb}_f$$

is the unique dimension-preserving unitary fiber functor up to a natural unitary monoidal isomorphism. Therefore the set $H^2(\widehat{SU_q(2)};\mathbb{T})$ consists of one element. The same results describe the dual second cohomology of free orthogonal quantum groups with coefficients in $\mathbb{T}$, and we see that typically this cohomology is nontrivial. □

It is natural to ask whether two cohomologous 2-cocycles that happen to be unitary, are in fact cohomologous as unitary cocycles, that is, if the canonical map $H^2(\hat{G};\mathbb{T}) \to H^2(\hat{G};\mathbb{C}^*)$ is injective. The following simple lemma settles this question in the affirmative.

Lemma 3.1.5. — *Let $\mathscr{E},\mathscr{F} \in \mathscr{U}(G\times G)$ be two unitary elements such that $\mathscr{E}=\mathscr{F}_u$ for an invertible element $u\in\mathscr{U}(G)$. Then also $\mathscr{E}=\mathscr{F}_v$, where v is the unitary part in the polar decomposition $u=v|u|$ of u.*

Proof. — It is sufficient to show that $(|u|\otimes|u|)\mathscr{F} = \mathscr{F}\hat\Delta(|u|)$, or equivalently, that $(u^*u\otimes u^*u)\mathscr{F}=\mathscr{F}\hat\Delta(u^*u)$. Since

$$1=\mathscr{E}^*\mathscr{E}=\hat\Delta(u^{-1})^*\mathscr{F}^*(u^*\otimes u^*)(u\otimes u)\mathscr{F}\hat\Delta(u^{-1}),$$

we see that this is indeed the case. □

A 2-cocycle $\mathscr{E}$ is invariant if it defines a natural morphism $U\otimes V \xrightarrow{\mathscr{E}^{-1}} U\otimes V$ in the tensor category $\operatorname{Rep} G$. In this case the identity functor $\operatorname{Rep} G\to\operatorname{Rep} G$ becomes a tensor functor with tensor structure given by the action of $\mathscr{E}^{-1}$. It follows that the group $H^2_G(\hat{G};\mathbb{C}^*)$ (resp. $H^2_G(\hat{G};\mathbb{T})$) can be identified with the group of all (resp. all unitary) monoidal autoequivalences of $\operatorname{Rep} G$ that are naturally isomorphic to the identity functor, considered up to natural (unitary) monoidal isomorphisms. Note that a functor $\operatorname{Rep} G\to\operatorname{Rep} G$ is naturally isomorphic to the identity functor if and only if it maps every irreducible representation into an equivalent one. Therefore we can say that the groups $H^2_G(\hat{G};\cdot)$ classify those monoidal autoequivalences of $\operatorname{Rep} G$ that define the trivial automorphism of the fusion ring of G.

Example 3.1.6. — Consider again the quantum group $SU_q(2)$, $q\in[-1,1]\setminus\{0\}$. As was shown in the proof of Theorem 2.5.11 (ii), any unitary monoidal autoequivalence of $SU_q(2)$ is naturally unitarily monoidally isomorphic to the identity functor. In particular, the group $H^2_{SU_q(2)}(\widehat{SU_q(2)};\mathbb{T})$ is trivial. □

One of the main goals of this chapter is to compute the groups $H^2_G(\hat{G};\mathbb{C}^*)$ and $H^2_G(\hat{G};\mathbb{T})$ for the Drinfeld-Jimbo deformations of arbitrary simply connected semisimple compact Lie groups. On the other hand, the computation of noninvariant second dual cohomology seems to be out of reach even for classical compact groups. We refer the reader to [**69**] for an overview of what is known in that case, but in order to get a taste of the problem let us briefly explain two known constructions of dual 2-cocycles.

Example 3.1.7. — Assume G is a compact quantum group and H is a closed quantum subgroup, that is, H is a compact quantum group and we have a surjective unital $*$-homomorphism $\pi\colon C(G)\to C(H)$ respecting the comultiplications. Then $\pi(\mathbb{C}[G])=\mathbb{C}[H]$, and by duality we get embeddings $\mathscr{U}(H^n)\hookrightarrow\mathscr{U}(G^n)$. Given a dual cocycle on H and using these embeddings we therefore get a dual cocycle on G. Such cocycles are said to be induced from H. In particular, if $H=\hat\Gamma$ for a discrete group Γ, then any 2-cocycle on Γ defines a dual 2-cocycle on G. $\square$

Example 3.1.8. — Assume G is a finite group and c is a $\mathbb{C}^*$-valued 2-cocycle on G. Assume c is nondegenerate, that is, for every element $g\in G\setminus\{e\}$ the character $h\mapsto c(g,h)c(h,g)^{-1}$ of the centralizer $C(g)$ of g is nontrivial. Consider the twisted group algebra A generated by elements u_g, $g\in G$, satisfying the relations $u_gu_h=c(g,h)u_{gh}$. The group G acts on A by the inner automorphisms $\operatorname{Ad}u_g$, so we get a representation of G on the vector space A. Using the nondegeneracy assumption it is easy to compute the character of this representation and conclude that the representation is equivalent to the regular one. Therefore we can identify the G-spaces A and $C(G)$, with G acting on $C(G)$ by right translations. The algebra structure on A defines a new product $\cdot$ on $C(G)$. Let $\tilde c(g,h)\in\mathbb{C}$ be such that

$$(a\cdot b)(e)=\sum_{g,h\in G}\tilde c(g,h)a(g)b(h)\ \text{ for all }\ a,b\in C(G).$$

Since the action of G respects the new product, we then get

$$(a\cdot b)(s)=\sum_{g,h\in G}\tilde c(g,h)a(gs)b(hs)\ \text{ for all }\ s\in G. \tag{3.1.1}$$

Define now an element $\mathscr{E}$ of $\mathscr{U}(G\times G)$ by

$$\mathscr{E}=\sum_{g,h\in G}\tilde c(g,h)\lambda_g\otimes\lambda_h,$$

where λ_g are the standard generators of the group algebra $\mathscr{U}(G)$ of G. Identity (3.1.1) can then be written as

$$(a\cdot b)(\omega)=(a\otimes b)(\mathscr{E}\hat\Delta(\omega))\ \text{ for all }\ \omega\in\mathscr{U}(G).$$

It follows that the associativity of the product $\cdot$ means exactly that $\mathscr{E}$ satisfies the cocycle identity

$$(\mathscr{E}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{E})=(1\otimes\mathscr{E})(\iota\otimes\hat\Delta)(\mathscr{E}).$$

It can furthermore be shown that $\mathscr{E}$ is invertible, so it is a dual 2-cocycle on G.

The construction of $\mathscr{E}$ depends on the identification of A with $C(G)$, but it is not difficult to see that the cohomology class of $\mathscr{E}$ depends only on the cohomology class of c. Therefore we get a map from the subset $H^2(G;\mathbb{C}^*)^\times \subset H^2(G;\mathbb{C}^*)$ of classes represented by nondegenerate 2-cocycles on G into $H^2(\hat{G};\mathbb{C}^*)$. $\square$

For finite groups any dual 2-cocycle is obtained by inducing a dual cocycle defined in the previous example from a subgroup [**31, 64, 92**]. The situation for general compact groups is much less clear.

3-cocycles

A 3-cocycle on $\hat{G}$ is an invertible element $\Phi \in \mathscr{U}(G^3)$ such that

$$(1\otimes\Phi)(\iota\otimes\hat{\Delta}\otimes\iota)(\Phi)(\Phi\otimes 1) = (\iota\otimes\iota\otimes\hat{\Delta})(\Phi)(\hat{\Delta}\otimes\iota\otimes\iota)(\Phi). \tag{3.1.2}$$

A 3-cocycle Φ' is cohomologous to Φ if there exists an invertible element $\mathscr{F} \in \mathscr{U}(G^2)$ such that

$$\Phi' = \Phi_{\mathscr{F}} = (1\otimes\mathscr{F})(\iota\otimes\hat{\Delta})(\mathscr{F})\Phi(\hat{\Delta}\otimes\iota)(\mathscr{F}^{-1})(\mathscr{F}^{-1}\otimes 1).$$

The relation of being cohomologous is not even symmetric. It nevertheless becomes an equivalence relation if we consider only invariant elements, that is, when Φ', Φ and $\mathscr{F}$ are invariant. Therefore we can define the cohomological sets $H^3_G(\hat{G};\mathbb{C}^*)$ and $H^3_G(\hat{G};\mathbb{T})$.

Lemma 3.1.9. — *Any (unitary) invariant 3-cocycle on $\hat{G}$ is cohomologous to a counital (unitary) invariant 3-cocycle.*

Proof. — Let Φ be a (unitary) invariant 3-cocycle. Consider the central (unitary) elements

$$u = (\iota\otimes\hat{\varepsilon}\otimes\hat{\varepsilon})(\Phi) \ \text{ and } \ v = (\hat{\varepsilon}\otimes\hat{\varepsilon}\otimes\iota)(\Phi)$$

in $\mathscr{U}(G)$, and put $\mathscr{F} = u\otimes v^{-1}$. We claim that $\Phi_{\mathscr{F}}$ is counital. Let us check, for example, that $(\hat{\varepsilon}\otimes\iota\otimes\iota)(\Phi_{\mathscr{F}}) = 1$, that is,

$$\hat{\Delta}(v^{-1})(\hat{\varepsilon}\otimes\iota\otimes\iota)(\Phi)(v\otimes 1) = 1.$$

By applying $\hat{\varepsilon}\otimes\hat{\varepsilon}\otimes\iota\otimes\iota$ to (3.1.2) we see that this is indeed the case. $\square$

Definition 3.1.10. — A counital unitary invariant 3-cocycle on $\hat{G}$ is called an associator for G.

The name associator is explained by the fact that any such element Φ defines new associativity morphisms $(U\otimes V)\otimes W \xrightarrow{\Phi} U\otimes(V\otimes W)$ on $\operatorname{Rep} G$. The cocycle condition (3.1.2) corresponds exactly to the pentagon diagram. For $G = \hat{\Gamma}$ this was already discussed in Example 2.1.2. Furthermore, two associators Φ and Φ' are cohomologous if and only if the C*-tensor categories $(\operatorname{Rep} G,\Phi)$ and $(\operatorname{Rep} G,\Phi')$ are unitarily monoidally equivalent via an equivalence that is naturally isomorphic to the identity functor. Therefore we can say that $H^3_G(\hat{G};\mathbb{T})$ classifies possible associativity morphisms on $\operatorname{Rep} G$ up to equivalence.

The set $H^3_G(\hat G;\mathbb{C}^*)$ has a similar interpretation, but then we have to go outside C*-tensor categories and allow for nonunitary associativity morphisms. Note also that similarly to the proof of Lemma 3.1.5 it is easy to check that the canonical map $H^3_G(\hat G;\mathbb{T}) \to H^3_G(\hat G;\mathbb{C}^*)$ is injective.

Example 3.1.11. — Consider the q-deformation G_q, $q > 0$, of a simply connected semisimple compact Lie group G. As we will discuss in detail in the next section, the center $Z(G)$ of G can be considered as a central subgroup of G_q, so every cocycle on $\widehat{Z(G)}$ defines an invariant dual cocycle on G_q. Explicitly, the dual group of $Z(G)$ is canonically isomorphic to P/Q, where P is the weight lattice and Q is the root lattice, and an n-cocycle c on P/Q considered as a cocycle on $\hat G_q$ acts on $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ as multiplication by $c(\lambda_1,\dots,\lambda_n)$.

Assume $G = SU(2)$ and $n = 3$. In this case $P = \frac12\mathbb{Z}$, $Q = \mathbb{Z}$, $P/Q \cong \mathbb{Z}/2\mathbb{Z}$. We have $H^3(\mathbb{Z}/2\mathbb{Z};\mathbb{T}) \cong \mathbb{Z}/2\mathbb{Z}$, with the cohomology class corresponding to 1 represented by the counital cocycle c such that $c(1,1,1) = -1$. This cocycle defines new associativity morphisms on $\operatorname{Rep} SU_q(2)$. By Theorem 2.5.10, if the category $\operatorname{Rep} SU_q(2)$ with new associativity morphisms has conjugates, then it must be unitarily monoidally equivalent to $\mathcal{TL}_{d,\tau}$ for some $d \ge 2$ and $\tau = \pm 1$. If a pair $(R,-R)$ solves the conjugate equations for $V_{1/2}$ in $\operatorname{Rep} SU_q(2)$, then (R,R) solves the conjugate equations for $V_{1/2}$ in the new nonstrict category. It follows that the new category is indeed a C*-tensor category with conjugates, and $d = \|R\|^2 = [2]_q$ and $\tau = -1$. Therefore, assuming that $q \in (0,1]$, the category $\operatorname{Rep} SU_q(2)$ with associativity morphisms defined by the cocycle c is unitarily monoidally equivalent to $\operatorname{Rep} SU_{-q}(2)$. □

In the next chapter we will prove much more: any of the categories $\operatorname{Rep} SU_q(2)$ can be obtained by changing the associativity morphisms in $\operatorname{Rep} SU(2)$, and the same is true for the q-deformation of any simply connected semisimple compact Lie group G.

We have therefore introduced cohomological sets/groups $H^1(\hat G;\cdot)$, $H^2(\hat G;\cdot)$, $H^1_G(\hat G;\cdot)$, $H^2_G(\hat G;\cdot)$ and $H^3_G(\hat G;\cdot)$, with coefficients in $\mathbb{T}$ or $\mathbb{C}^*$. They all have categorical interpretations, and $H^i_G(\hat G;\cdot)$ depend only on the C*-tensor category $\operatorname{Rep} G$.

References. — [**14**], [**19**], [**20**], [**27**], [**31**], [**38**], [**43**], [**42**], [**57**], [**64**], [**69**], [**92**].

3.2. GROUP-LIKE ELEMENTS

In this section we will compute the groups $H^1(\hat G;\cdot)$ and $H^1_G(\hat G;\cdot)$ in various cases. We start with genuine compact groups.

Theorem 3.2.1. — *For any compact group G we have canonical isomorphisms*

$$H^1(\hat G;\mathbb{T}) \cong G \ \ \textit{and} \ \ H^1_G(\hat G;\mathbb{T}) \cong Z(G),$$

where $Z(G)$ is the center of G.

Proof. — Elements of G define evaluations maps on $\mathbb{C}[G]$ that are characters. We have to show that this way we get all characters of $\mathbb{C}[G]$. There are several ways of proving this. One possibility is to use coamenability of G established in Proposition 2.7.12. It implies that any character χ of $\mathbb{C}[G]$ extends to $C(G) = C_u(G)$, hence χ is the evaluation at some element of G. Therefore $H^1(\hat{G};\mathbb{T}) \cong G$.

This result can also be formulated as follows. The algebra $\mathscr{U}(G)$ contains the group algebra of G, generated by the elements λ_g such that $\pi_U(\lambda_g) = U_g$ for any unitary representation U. Then the set of unitary group-like elements in $\mathscr{U}(G)$ is exactly $\{\lambda_g\}_{g\in G}$.

Clearly, λ_g is central in $\mathscr{U}(G)$ if and only if g is central in G. Hence $H^1_G(\hat{G};\mathbb{T}) \cong Z(G)$. □

For compact Lie groups we can also describe the set of all group-like elements. In order to do this, recall that the analytic complexification $G_{\mathbb{C}}$ of a compact Lie group G is a complex analytic group containing G such that any Lie group homomorphism $G \to H$ into a complex analytic group H extends uniquely to an analytic homomorphism $G_{\mathbb{C}} \to H$, see e.g., [**17**, Chapter 27]. Any compact Lie group has an analytic complexification, and it is unique up to an isomorphism. Furthermore, the Lie algebra of $G_{\mathbb{C}}$ is the complexification of the Lie algebra of G, and $G_{\mathbb{C}}/(G_{\mathbb{C}})^\circ = G/G^\circ$. Since any finite dimensional representation of G extends to a representation of $G_{\mathbb{C}}$ on the same space, we have an embedding $G_{\mathbb{C}} \hookrightarrow \mathscr{U}(G)$. Its image consists of group-like elements.

Theorem 3.2.2. — *For any compact Lie group G we have canonical isomorphisms*

$$H^1(\hat{G};\mathbb{C}^*) \cong G_{\mathbb{C}} \ \text{ and } \ H^1_G(\hat{G};\mathbb{C}^*) \cong Z(G_{\mathbb{C}}).$$

Proof. — Assume a is a group-like element in $\mathscr{U}(G)$. Then a^*a is group-like as well, hence $|a|$ is also group-like. It follows that if $a = u|a|$ is the polar decomposition, then u is group-like. By the previous theorem we know that $u \in G$. So we just have to show that $|a| \in G_{\mathbb{C}}$. In other words, we may assume that a is positive.

For every $z \in \mathbb{C}$ we have

$$\hat{\Delta}(a^z) = \hat{\Delta}(a)^z = (a \otimes a)^z = a^z \otimes a^z.$$

In particular, the unitary elements a^{it}, $t \in \mathbb{R}$, are group-like, hence they lie in $G \subset \mathscr{U}(G)$. It follows that there exists an element X of the Lie algebra of G such that $a^{it} = \exp tX$ for $t \in \mathbb{R}$. Then $a^z = \exp(-izX) \in G_{\mathbb{C}}$ for all $z \in \mathbb{C}$, since both a^z and $\exp(-izX)$ are analytic functions in z that coincide for $z \in i\mathbb{R}$. In particular, $a = \exp(-iX) \in G_{\mathbb{C}}$. □

Consider now the q-deformation G_q of a simply connected semisimple compact Lie group G, $q > 0$. The irreducible $*$-representations of $\mathbb{C}[G_q]$ for $q \neq 1$ have been classified by Soibelman, see [**53**]. According to this classification the characters of $\mathbb{C}[G_q]$ correspond to points of our fixed maximal torus $T \subset G$. Explicitly, we embed T into $\mathscr{U}(G_q)$ by

$$\exp(it_1(\log q_1)H_1 + \cdots + it_r(\log q_r)H_r) \mapsto K_1^{it_1} \dots K_r^{it_r},$$

where $H_1, \dots, H_r$ is the standard basis in $\mathfrak{h}$. Note that this embedding agrees with our conventions in Example 2.6.4, where we introduced elements H_j of $\mathscr{U}(G_q)$ such that $K_j = q_j^{H_j}$.

Theorem 3.2.3. — *For every $q > 0$, $q \neq 1$, we have:*
(i) $H^1(\hat{G}_q; \mathbb{T}) = T$ and $H^1(\hat{G}_q; \mathbb{C}^*) = T_{\mathbb{C}}$; in other words, the group-like elements in $\mathscr{U}(G_q)$ have the form $K_1^{z_1} \dots K_r^{z_r}$, with $z_1, \dots, z_r \in \mathbb{C}$;
(ii) $H^1_{G_q}(\hat{G}_q; \mathbb{T}) = H^1_{G_q}(\hat{G}_q; \mathbb{C}^*) = Z(G) \cong \widehat{P/Q}$.

Proof. — (i) As we already explained, the equality $H^1(\hat{G}_q; \mathbb{T}) = T$ is part of Soibelman's classification of irreducible representations of $\mathbb{C}[G_q]$. The equality $H^1(\hat{G}_q; \mathbb{C}^*) = T_{\mathbb{C}}$ is then obtained by the same argument as in the proof of Theorem 3.2.2.

(ii) By considering $T_{\mathbb{C}}$ as a subset of $\mathscr{U}(G_q)$, from the relations in $U_q\mathfrak{g}$ we get

$$\exp(H)E_j\exp(-H) = e^{\alpha_j(H)}E_j \quad \text{for any} \quad H \in \mathfrak{h}.$$

It follows that an element $\exp(H)$ is central in $\mathscr{U}(G_q)$ if and only if $e^{\alpha_j(H)} = 1$ for all j. Identifying $T_{\mathbb{C}}$ with the group of quasi-characters of the weight lattice P (namely, the image of $\lambda \in P$ under $\exp(H)$ is $e^{\lambda(H)}$), we conclude that an element of $T_{\mathbb{C}}$ lies in the center of $\mathscr{U}(G_q)$ if and only if it vanishes on the root lattice Q. Since P/Q is finite, any quasi-character of P/Q is a character. Therefore

$$H^1_{G_q}(\hat{G}_q; \mathbb{T}) = H^1_{G_q}(\hat{G}_q; \mathbb{C}^*) \cong \widehat{P/Q}.$$

Finally, it is well-known that the center $Z(G)$ of G is contained in T and it consists exactly of those characters of P that vanish on Q. □

Part (ii) of the above theorem can, in fact, be easily proved without relying on the classification of irreducible representations of $\mathbb{C}[G_q]$. Since the irreducible representations of G_q are classified by dominant integral weights, the center of $\mathscr{U}(G_q)$ is canonically isomorphic to the algebra of functions on P_+. In particular, the centers of $\mathscr{U}(G_q)$ and $\mathscr{U}(G)$ are canonically isomorphic.

Lemma 3.2.4. — *There exists a $*$-isomorphism $\varphi\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ that extends the canonical identification of the centers and that is the identity map on the maximal torus T. Furthermore, there exists a unitary $\mathscr{F} \in \mathscr{U}(G \times G)$ such that $(\varphi \otimes \varphi)\hat{\Delta}_q = \mathscr{F}\hat{\Delta}\varphi(\cdot)\mathscr{F}^*$.*

Proof. — For every $\lambda \in P_+$ fix a representation V^q_λ of G_q and a representation V_λ of G with highest weights λ. Then to define φ is the same as to choose a $*$-isomorphism $\varphi_\lambda\colon B(V^q_\lambda) \to B(V_\lambda)$ that maps the projection onto a weight space $V^q_\lambda(\mu)$ into the projection onto $V_\lambda(\mu)$ for all $\lambda \in P_+$ and $\mu \in P$. Since by Theorem 2.4.7 the dimensions of V^q_λ and $V^q_\lambda(\mu)$ do not depend on q, such an isomorphism clearly exists. Next, by the same theorem, the multiplicity of V^q_η in $V^q_\lambda \otimes V^q_\nu$ does not depend on q either. This means that the multiplicity of the map $\hat\Delta^\eta_{q;\lambda,\nu}\colon B(V^q_\eta) \to B(V^q_\lambda) \otimes B(V^q_\nu)$ obtained by applying $\hat\Delta_q$ to an element of $B(V^q_\eta) \subset \mathscr{U}(G_q)$ and projecting the image onto $B(V^q_\lambda) \otimes B(V^q_\nu)$, does not depend on q. It follows that the two maps

$$\hat\Delta^\eta_{\lambda,\nu},\ (\varphi_\lambda \otimes \varphi_\nu)\hat\Delta^\eta_{q;\lambda,\nu}\varphi^{-1}_\eta\colon B(V_\eta) \to B(V_\lambda) \otimes B(V_\nu)$$

have the same multiplicities, hence they are inner conjugate. This implies the existence of $\mathscr{F}$. $\square$

The existence of $(\varphi, \mathscr{F})$ implies that a central element in $\mathscr{U}(G_q)$ is group-like if and only if it is group-like in $\mathscr{U}(G)$. Indeed, if, for example, c is group-like in $\mathscr{U}(G)$, then

$$(\varphi \otimes \varphi)\hat\Delta_q(c) = \mathscr{F}\hat\Delta(c)\mathscr{F}^* = c \otimes c,$$

so c is group-like in $\mathscr{U}(G_q)$ as well. Hence

$$H^1_{G_q}(\hat G_q; \mathbb{C}^*) = H^1_G(\hat G; \mathbb{C}^*) = Z(G_{\mathbb{C}}) = Z(G).$$

Finally, note that a central element $c \in \mathscr{U}(G_q)$ is simply a collection of numbers $c(\lambda)$, $\lambda \in P_+$, such that c acts on V^q_λ as multiplication by $c(\lambda)$. Then c is group-like if and only if $c(\lambda) = \chi(\lambda)$ for a character χ of P/Q. In this formulation it is particularly obvious that $\mathscr{U}(G_q)$ has no nontrivial positive central group-like elements, as we claimed in the proof of Proposition 2.4.10.

References. — [**17**], [**29**], [**53**], [**70**].

3.3. KAZHDAN-LUSZTIG COMONOID

In this section we will introduce a construction that will play a crucial role in the analysis of invariant dual 2-and 3-cocycles.

Let M be any object in a C*-tensor category $\mathscr{C}$ with associativity morphisms $\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$. Assume that λ and ρ are the identity morphisms, so that $\mathbb{1} \otimes U = U \otimes \mathbb{1} = U$. Consider the functor $F = \mathrm{Hom}(M, \cdot)$ from $\mathscr{C}$ into the category of vector spaces. In general there is no obvious way to define a tensor structure on F.

Definition 3.3.1. — An object M in $\mathscr{C}$ is called a comonoid if it comes with two morphisms $\delta\colon M \to M \otimes M$ and $\varepsilon\colon M \to \mathbb{1}$ such that

$$\alpha(\delta \otimes \iota)\delta = (\iota \otimes \delta)\delta, \quad (\varepsilon \otimes \iota)\delta = (\iota \otimes \varepsilon)\delta = \iota.$$

Given a comonoid M and letting $F = \mathrm{Hom}(M, \cdot)$ we can define natural morphisms

$$F_2\colon F(U) \otimes F(V) \to F(U \otimes V)$$

by $f \otimes g \mapsto (f \otimes g)\delta$. If these morphisms happen to be isomorphisms and the spaces $F(M)$ are finite dimensional, then (F, F_2) becomes a fiber functor, with $F_0\colon \mathbb{C} \to F(\mathbb{1})$ defined by $1 \mapsto \varepsilon$.

Example 3.3.2. — Let G be a finite quantum group, that is, $C(G)$ is finite dimensional. Then the triple $(\mathscr{U}(G), \hat{\Delta}, \hat{\varepsilon})$ is a comonoid, called the regular comonoid. The corresponding fiber functor $F = \mathrm{Hom}(\mathscr{U}(G), \cdot)$ is naturally monoidally isomorphic to the canonical fiber functor, with the isomorphisms

$$\mathrm{Hom}(\mathscr{U}(G), U) \simeq U, \quad f \mapsto f(1).$$

More generally, if $\mathscr{E}$ is a counital 2-cocycle on $\hat{G}$, we can define a map

$$\delta\colon \mathscr{U}(G) \to \mathscr{U}(G \times G), \quad \delta(a) = \hat{\Delta}(a)\mathscr{E}^{-1}.$$

Again it is easy to see that $(\mathscr{U}(G), \delta, \hat{\varepsilon})$ is a comonoid in $\mathrm{Rep}\, G$. It represents the forgetful functor, but now with the tensor structure associated to the cocycle $\mathscr{E}$, that is,

$$F_2 = \mathscr{E}^{-1}\colon U \otimes V \to U \otimes V.$$

□

We now focus on the C*-tensor category $\mathscr{C}_q(\mathfrak{g})$, $q > 0$, of finite dimensional admissible unitary $U_q\mathfrak{g}$-modules introduced in Section 2.4. The aim of this section is to construct, rather elaborately, a comonoid M representing the canonical fiber functor.

Our comonoid cannot live inside $\mathscr{C}_q(\mathfrak{g})$, since $\mathscr{C}_q(\mathfrak{g})$ has infinitely many nonisomorphic simple objects, so we have to enlarge the category $\mathscr{C}_q(\mathfrak{g})$, namely, to the category pro-$\mathscr{C}_q(\mathfrak{g})$ of pro-objects. This means that the objects of the new category are formal cofiltered limits, or in other words projective systems $\{V_i\}_i$ of finite dimensional admissible unitary $U_q\mathfrak{g}$-modules, and

$$\mathrm{Hom}(\{V_i\}_i, \{W_j\}_j) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{U_q\mathfrak{g}}(V_i, W_j).$$

It is convenient, although not fully correct, to think of a pro-object $\{V_i\}_i$ as a topological $U_q\mathfrak{g}$-module $V = \varprojlim V_i$, with a base of neighborhoods of zero formed by the kernels of the canonical morphisms $V \to V_i$; in particular, finite dimensional admissible $U_q\mathfrak{g}$-modules are considered with discrete topology. If the morphisms $V \to V_i$ are surjective, then for any finite dimensional admissible $U_q\mathfrak{g}$-module W any continuous $U_q\mathfrak{g}$-module map $V \to W$ factors through V_i for some i, so the space of such maps is the inductive limit of $\mathrm{Hom}_{U_q\mathfrak{g}}(V_i, W)$, as required.

The category pro-$\mathscr{C}_q(\mathfrak{g})$ is monoidal, with the tensor product defined by

$$\{V_i\}_i \otimes \{W_j\}_j = \{V_i \otimes W_j\}_{i,j},$$

but it is not a C*-tensor category.

Returning to the construction of a representing object, the simplest possibility is to take $\mathscr{U}(G_q)$. But our module M will have the additional important property that the left action of $U_q\mathfrak{g}$ extends to an action of $\mathscr{U}(G_q\times G_q)$ under the embedding $\hat\Delta_q\colon U_q\mathfrak{g}\to\mathscr{U}(G_q\times G_q)$. A way to think of M is as a completion of $U_q(\mathfrak{g}\oplus\mathfrak{h})$, with the action of $U_q\mathfrak{g}\otimes U_q\mathfrak{g}$ defined using the Poincare-Birkhoff-Witt decomposition $U_q(\mathfrak{g}\oplus\mathfrak{h})=U_q\mathfrak{b}_-\otimes U_q\mathfrak{b}_+$, where the action of $U_q(\mathfrak{g}\oplus\mathfrak{h})$ on $U_q\mathfrak{b}_\pm$ is defined by identifying the latter space with the induced module $\mathrm{Ind}^{U_q(\mathfrak{g}\oplus\mathfrak{h})}_{U_q\mathfrak{b}_\mp}\hat\varepsilon_q$.

The construction of M is based on the following representation-theoretic result, see e.g., [**60**, Proposition 23.3.10]. Recall first that for a unitary $U_q\mathfrak{g}$-module V we denote by $\bar V$ the conjugate module, with the action of $U_q\mathfrak{g}$ given by $\omega\bar\xi=\overline{\hat R_q(\omega)^*\xi}$. For the highest weight vector $\xi_\lambda\in V_\lambda$, the vector $\bar\xi_\lambda$ is a lowest weight vector of weight $-\lambda$.

Proposition 3.3.3. — *For any finite dimensional admissible $U_q\mathfrak{g}$-module V and $\lambda\in P$, the map*

$$\mathrm{Hom}_{U_q\mathfrak{g}}(\bar V_\mu\otimes V_{\lambda+\mu},V)\to V(\lambda),\quad f\mapsto f(\bar\xi_\mu\otimes\xi_{\lambda+\mu}),$$

is an isomorphism for all sufficiently large weights $\mu\in P_+$.

The modules $\bar V_\mu\otimes V_{\lambda+\mu}$ form a projective system in a natural way.

Lemma 3.3.4. — *For any $\lambda\in P$ and $\mu,\eta\in P_+$ such that $\lambda+\mu\in P_+$, there exists a unique morphism*

$$\mathrm{tr}^\eta_{\mu,\lambda+\mu}\colon \bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\to\bar V_\mu\otimes V_{\lambda+\mu}$$

such that $\bar\xi_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta}\mapsto\bar\xi_\mu\otimes\xi_{\lambda+\mu}$.

Proof. — The uniqueness is clear, since the vector $\bar\xi_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta}$ is cyclic.

In order to show the existence, we introduce auxiliary maps. For $\mu,\eta\in P_+$, the vector $\xi_\mu\otimes\xi_\eta$ is killed by the E_i's and has weight $\mu+\eta$. Hence we can define a morphism

$$T_{\mu,\eta}\colon V_{\mu+\eta}\to V_\mu\otimes V_\eta\ \text{ such that }\ \xi_{\mu+\eta}\mapsto\xi_\mu\otimes\xi_\eta.$$

Similarly define a morphism

$$\bar T_{\mu,\eta}\colon \bar V_{\mu+\eta}\to\bar V_\mu\otimes\bar V_\eta\ \text{ such that }\ \bar\xi_{\mu+\eta}\mapsto\bar\xi_\mu\otimes\bar\xi_\eta.$$

In addition, since $\bar V_\mu$ is conjugate to V_μ, there exists a unique up to a scalar factor nonzero morphism

$$S_\mu\colon\bar V_\mu\otimes V_\mu\to V_0=\mathbb{C}.$$

We normalize it so that $S_\mu(\bar\xi_\mu\otimes\xi_\mu)=1$.

Then the required map $\mathrm{tr}^\eta_{\mu,\lambda+\mu}$ can be written as the composition

$$\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\xrightarrow{\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}}\bar V_\mu\otimes\bar V_\eta\otimes V_\eta\otimes V_{\lambda+\mu}\xrightarrow{\iota\otimes S_\eta\otimes\iota}\bar V_\mu\otimes V_{\lambda+\mu}.\qquad\square$$

Denote by M_λ the pro-object defined by the projective system $\{\bar V_\mu \otimes V_{\lambda+\mu}\}_\mu$. By Proposition 3.3.3 it represents the functor $\mathscr{C}_q(\mathfrak{g}) \to \mathrm{Hilb}_f$, $V \mapsto V(\lambda)$, so we have natural isomorphisms

$$\mathrm{Hom}(M_\lambda, V) \cong V(\lambda).$$

These can also be described as follows. By slightly abusing notation denote by the same symbol M_λ the topological $U_q\mathfrak{g}$-module $\varprojlim \bar V_\mu \otimes V_{\lambda+\mu}$. Let $\Omega_\lambda \in M_\lambda$ be the limit vector of weight λ defined by the cyclic vectors $\bar\xi_\mu \otimes \xi_{\lambda+\mu}$. Then Ω_λ is a topologically cyclic vector for M_λ, and the isomorphism $\mathrm{Hom}_{U_q\mathfrak{g}}(M_\lambda, V) \cong V(\lambda)$ is given by $f \mapsto f(\Omega_\lambda)$.

We are now ready to define our pro-object representing the forgetful functor:

$$M = \left\{ \bigoplus_{\lambda\in X} \bar V_\mu \otimes V_{\lambda+\mu} \right\}_{X,\mu},$$

where the index set consists of pairs (X, μ) such that $X \subset P$ is finite and $\mu \in P_+$ is such that $\lambda + \mu \in P_+$ for all $\lambda \in X$. The maps $\bigoplus_{\lambda\in X} \bar V_\mu \otimes V_{\lambda+\mu} \to \bigoplus_{\lambda\in Y} \bar V_\eta \otimes V_{\lambda+\eta}$ are defined for $Y \subset X$ and $\mu - \eta \in P_+$ as the composition of the projection

$$\bigoplus_{\lambda\in X} \bar V_\mu \otimes V_{\lambda+\mu} \to \bigoplus_{\lambda\in Y} \bar V_\mu \otimes V_{\lambda+\mu}$$

with the map $\bigoplus_{\lambda\in Y} \bar V_\mu \otimes V_{\lambda+\mu} \to \bigoplus_{\lambda\in Y} \bar V_\eta \otimes V_{\lambda+\eta}$ defined by the maps $\mathrm{tr}^{\mu-\eta}_{\eta,\lambda+\eta}$.

If we think of M as a topological $U_q\mathfrak{g}$-module, then

$$M = \prod_{\lambda\in P} M_\lambda.$$

Due to the weight decomposition of V, the pro-object M indeed represents the forgetful functor $\mathscr{C}_q(\mathfrak{g}) \to \mathrm{Hilb}_f$. Let

$$\eta_V\colon \mathrm{Hom}(M, V) \to V$$

be the canonical isomorphism of these two functors. Again, if we view M as a topological $U_q\mathfrak{g}$-module, then this isomorphism can be described as follows. The vector $\Omega = (\Omega_\lambda)_\lambda \in \prod_\lambda M_\lambda = M$ is topologically cyclic, and

$$\eta_V(f) = f(\Omega) \ \text{ for } \ f \in \mathrm{Hom}_{U_q\mathfrak{g}}(M, V).$$

Our next goal is to define a comonoid structure on M. On the level of modules the formula is straightforward to guess. First of all note that the pro-object $M \otimes M$ considered as a topological module is

$$\prod_{\lambda_1,\lambda_2\in P} M_{\lambda_1} \otimes M_{\lambda_2},$$

where

$$M_{\lambda_1} \otimes M_{\lambda_2} = \varprojlim_{\mu_1,\mu_2} (\bar V_{\mu_1} \otimes V_{\lambda_1+\mu_1}) \otimes (\bar V_{\mu_2} \otimes V_{\lambda_2+\mu_2}).$$

Lemma 3.3.5. — *There exists a unique morphism* $\delta\colon M \to M \otimes M$ *such that on the level of modules we have* $\delta(\Omega) = \Omega \otimes \Omega = (\Omega_{\lambda_1} \otimes \Omega_{\lambda_2})_{\lambda_1,\lambda_2}$.

Proof. — The uniqueness is clear, since Ω is a topologically cyclic vector. In order to prove the existence note that since δ should preserve the weight decompositions, we must have

$$\delta(\Omega_\lambda) = (\Omega_{\lambda_1} \otimes \Omega_{\lambda_2})_{\lambda_1+\lambda_2=\lambda}.$$

Therefore we have to show that for all $\lambda_1, \lambda_2 \in P$ there exists a morphism

$$\delta_{\lambda_1,\lambda_2}\colon M_{\lambda_1+\lambda_2} \to M_{\lambda_1} \otimes M_{\lambda_2} \text{ such that } \delta_{\lambda_1,\lambda_2}(\Omega_{\lambda_1+\lambda_2}) = \Omega_{\lambda_1} \otimes \Omega_{\lambda_2}.$$

In order to define $\delta_{\lambda_1,\lambda_2}$ it suffices to show that there exist morphisms

$$m_{\mu,\eta,\lambda_1,\lambda_2}\colon \bar{V}_{\mu+\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda_1+\mu} \otimes \bar{V}_\eta \otimes V_{\lambda_2+\eta}$$

that map $\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda_1+\lambda_2+\mu+\eta}$ into $\bar{\xi}_\mu \otimes \xi_{\lambda_1+\mu} \otimes \bar{\xi}_\eta \otimes \xi_{\lambda_2+\eta}$. Since $\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda_1+\lambda_2+\mu+\eta}$ is a cyclic vector, the morphism m is unique, if it exists. In order to show the existence, recall first that the braiding $\sigma = \Sigma\mathscr{R}_q$ in $\mathscr{C}_q(\mathfrak{g})$ is given by

$$\sigma\colon \bar{V}_\mu \otimes V_\eta \to V_\eta \otimes \bar{V}_\mu, \quad \bar{\xi}_\mu \otimes \xi_\eta \mapsto q^{-(\mu,\eta)}\xi_\eta \otimes \bar{\xi}_\mu,$$

and then write m as the composition

$$\begin{aligned}\bar{V}_{\mu+\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} &\xrightarrow{\bar{T}_{\mu,\eta}\otimes T_{\lambda_1+\mu,\lambda_2+\eta}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_{\lambda_1+\mu} \otimes V_{\lambda_2+\eta} \\ &\xrightarrow{q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)} \bar{V}_\mu \otimes V_{\lambda_1+\mu} \otimes \bar{V}_\eta \otimes V_{\lambda_2+\eta}.\end{aligned}$$

□

We also introduce a morphism $\varepsilon\colon M \to \mathbb{C}$ by $\varepsilon(\Omega) = 1$. In other words, ε is determined by the morphisms

$$\mathrm{tr}^\mu_{0,0} = S_\mu\colon \bar{V}_\mu \otimes V_\mu \to \mathbb{C}.$$

It is straightforward to check that (M, δ, ε) is a comonoid. Since $\eta_V(f) = f(\Omega)$, by definition of δ we also immediately get $\eta_{U\otimes V}((f \otimes g)\delta) = \eta_U(f) \otimes \eta_V(g)$. Thus we obtain the following result.

Theorem 3.3.6. — *The triple* (M, δ, ε) *is a comonoid in* pro-$\mathscr{C}_q(\mathfrak{g})$ *representing the canonical fiber functor* $\mathscr{C}_q(\mathfrak{g}) \to \mathrm{Hilb}_f$.

Just as the regular comonoid $\mathscr{U}(G)$ is a $\mathscr{U}(G)$-bimodule, M carries a right action of $U_q\mathfrak{g}$. In other words, the algebra $(U_q\mathfrak{g})^{\mathrm{op}}$ with the opposite multiplication acts by endomorphisms on the pro-object M. For an element $X \in U_q\mathfrak{g}$ denote by $\tilde{X}$ the same element considered as an element of $(U_q\mathfrak{g})^{\mathrm{op}}$.

Theorem 3.3.7. — *There exists a unique representation* $(U_q\mathfrak{g})^{\mathrm{op}} \to \mathrm{End}(M)$ *such that on the level of modules we have* $\tilde{X}\Omega = X\Omega$ *for all* $X \in U_q\mathfrak{g}$.

Proof. — The uniqueness is again clear. For the proof of the existence we will assume that $q \neq 1$, the case $q = 1$ requires only minor, mostly notational, changes.

Since $\tilde{K}_i$ should preserve the weight components of M, we must have $\tilde{K}_i\Omega_\lambda = q_i^{\lambda(i)}\Omega_\lambda = K_i\Omega_\lambda$. Thus we let $\tilde{K}_i$ act on M_λ by the scalar $q_i^{\lambda(i)}$. For the same reason we must have

$$\tilde{F}_i\Omega_\lambda = F_i\Omega_{\lambda+\alpha_i} \text{ and } \tilde{E}_i\Omega_\lambda = E_i\Omega_{\lambda-\alpha_i}.$$

To prove the theorem it suffices to show that such morphisms $M_\lambda \to M_{\lambda+\alpha_i}$ and $M_\lambda \to M_{\lambda-\alpha_i}$ indeed exist, since then we get $\tilde{X}_n \dots \tilde{X}_1\Omega = X_1 \dots X_n\Omega$ for all $X_1, \dots, X_n \in \{F_i, E_i, K_i, K_i^{-1}\}$, which by the topological cyclicity of Ω shows that any relation in $(U_q\mathfrak{g})^{\mathrm{op}}$ is satisfied by the endomorphisms $\tilde{F}_i, \tilde{E}_i, \tilde{K}_i, \tilde{K}_i^{-1}$ of M.

In order to define $\tilde{F}_i\colon M_\lambda \to M_{\lambda+\alpha_i}$ it suffices to show that for sufficiently large μ and η there exist morphisms

$$\Psi^\eta_{i;\mu,\lambda+\alpha_i+\mu}\colon \bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu}$$

such that

$$\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta} \mapsto \hat{\Delta}_q(F_i)(\bar{\xi}_\mu \otimes \xi_{\lambda+\alpha_i+\mu}) = \bar{\xi}_\mu \otimes F_i\xi_{\lambda+\alpha_i+\mu}.$$

For this, in turn, consider $\mu, \eta \in P_+$ such that $\mu(i), \eta(i) \geq 1$. Then the space $(V_\mu \otimes V_\eta)(\mu + \eta - \alpha_i)$ is 2-dimensional, spanned by $F_i\xi_\mu \otimes \xi_\eta$ and $\xi_\mu \otimes F_i\xi_\eta$. As one can easily check, this space has a unique up to a scalar factor vector killed by E_i, namely,

$$[\mu(i)]_{q_i}\xi_\mu \otimes F_i\xi_\eta - q_i^{\mu(i)}[\eta(i)]_{q_i}F_i\xi_\mu \otimes \xi_\eta.$$

In other words, the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta - \alpha_i$ is the image of the morphism

$$\tau_{i;\mu,\eta}\colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta$$

such that

$$\xi_{\mu+\eta-\alpha_i} \mapsto [\mu(i)]_{q_i}\xi_\mu \otimes F_i\xi_\eta - q_i^{\mu(i)}[\eta(i)]_{q_i}F_i\xi_\mu \otimes \xi_\eta.$$

The morphism $\Psi^\eta_{i;\mu,\lambda+\alpha_i+\mu}$ can now be written as the composition

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{T}_{\mu,\eta}\otimes\tau_{i;\eta,\lambda+\alpha_i+\mu}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\alpha_i+\mu} \xrightarrow{\iota\otimes S_\eta\otimes\iota} \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu}.$$

Similarly, consider the morphisms

$$\bar{\tau}_{i;\mu,\eta}\colon \bar{V}_{\mu+\eta-\alpha_i} \to \bar{V}_\mu \otimes \bar{V}_\eta$$

such that

$$\bar{\xi}_{\mu+\eta-\alpha_i} \mapsto [\eta(i)]_{q_i}E_i\bar{\xi}_\mu \otimes \bar{\xi}_\eta - q_i^{\eta(i)}[\mu(i)]_{q_i}\bar{\xi}_\mu \otimes E_i\bar{\xi}_\eta.$$

Then $\tilde{E}_i\colon M_\lambda \to M_{\lambda-\alpha_i}$ is defined using the morphisms

$$\Phi^\eta_{i;\mu+\alpha_i,\lambda+\mu}\colon \bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \to \bar{V}_{\mu+\alpha_i} \otimes V_{\lambda+\mu}$$

such that

$$\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta} \mapsto \hat{\Delta}_q(E_i)(\bar{\xi}_{\mu+\alpha_i} \otimes \xi_{\lambda+\mu}) = E_i\bar{\xi}_{\mu+\alpha_i} \otimes \xi_{\lambda+\mu},$$

which are well-defined as they can be written as the compositions

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{\tau}_{i;\mu+\alpha_i,\eta}\otimes T_{\eta,\lambda+\mu}} \bar{V}_{\mu+\alpha_i} \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\mu} \xrightarrow{\iota\otimes S_\eta\otimes\iota} \bar{V}_{\mu+\alpha_i} \otimes V_{\lambda+\mu}. \qquad \square$$

The action of $(U_q\mathfrak{g})^{\mathrm{op}}$ allows us to consider the functor $\mathrm{Hom}(M,\cdot)$ as a functor from $\mathscr{C}_q(\mathfrak{g})$ into the category of finite dimensional admissible $U_q\mathfrak{g}$-modules: the action of $U_q\mathfrak{g}$ on $\mathrm{Hom}(M,V)$ is defined by $Xf = f\tilde{X}$. Furthermore, it is immediate that

$$\delta\tilde{X} = \hat{\Delta}_q(\tilde{X})\delta \ \text{ and } \ \varepsilon\tilde{X} = \hat{\varepsilon}_q(X)\varepsilon$$

for all $X \in U_q\mathfrak{g}$. Hence $\mathrm{Hom}(M,\cdot)$ is a tensor functor. This functor is monoidally isomorphic, via the natural isomorphisms η_V, to the identity functor on $\mathscr{C}_q(\mathfrak{g})$.

References. — [**33**], [**49**], [**60**], [**70**].

3.4. COMPUTATION OF INVARIANT SECOND COHOMOLOGY

Let G_q, $q>0$, be the q-deformation of a simply connected semisimple compact Lie group G. As follows from the discussion in Section 3.2, the center $Z(G)$ of G can be considered as a subgroup of the quantum group G_q. Therefore, as was already used in Example 3.1.11, any $\mathbb{T}$-valued dual 2-cocycle on $Z(G)$ can be induced to a dual 2-cocycle on G_q. More explicitly, a dual 2-cocycle on $Z(G)$ is a 2-cocycle c on $\widehat{Z(G)} = P/Q$. Then the induced dual cocycle $\mathscr{E}_c$ on G_q acts on $V_\mu \otimes V_\eta$ as multiplication by $c(\mu,\eta)$. This cocycle is obviously invariant. It turns out that this way we get all invariant cocycles up to coboundaries.

Theorem 3.4.1. — *The homomorphism $c \mapsto \mathscr{E}_c$ induces an isomorphism*

$$H^2(P/Q;\mathbb{T}) \cong H^2_{G_q}(\hat{G}_q;\mathbb{T}) = H^2_{G_q}(\hat{G}_q;\mathbb{C}^*).$$

Before we turn to the proof, let us see what the result means for simple groups.

Corollary 3.4.2. — *If $\mathfrak{g}$ is simple and $\mathfrak{g} \ncong \mathfrak{so}_{4n}(\mathbb{C})$ for $n \ge 1$, then the groups $H^2_{G_q}(\hat{G}_q;\mathbb{T})$ and $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$ are trivial, and if $\mathfrak{g} \cong \mathfrak{so}_{4n}(\mathbb{C})$, then $H^2_{G_q}(\hat{G}_q;\mathbb{T}) = H^2_{G_q}(\hat{G}_q;\mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. — For simple Lie algebras the group P/Q is cyclic unless $\mathfrak{g} \cong \mathfrak{so}_{4n}(\mathbb{C})$, in which case $P/Q \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, see e.g., Table IV on page 516 in [**39**]. $\square$

Turning to the proof of the theorem, note that it suffices to show that the map $c \mapsto \mathscr{E}_c$ induces an isomorphism $H^2(P/Q;\mathbb{C}^*) \cong H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$. Indeed, since P/Q is finite, the map $H^2(P/Q;\mathbb{T}) \to H^2(P/Q;\mathbb{C}^*)$ is an isomorphism, so we conclude that any $\mathbb{C}^*$-valued invariant dual 2-cocycle on G_q is represented by a unitary cocycle. At the same time the map $H^2_{G_q}(\hat{G}_q;\mathbb{T}) \to H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$ is injective by Lemma 3.1.5. Therefore we get $H^2_{G_q}(\hat{G}_q;\mathbb{T}) \cong H^2_{G_q}(\hat{G}_q;\mathbb{C}^*) \cong H^2(P/Q;\mathbb{T})$.

Our first goal will be to construct a homomorphism

$$H^2_{G_q}(\hat{G}_q;\mathbb{C}^*) \to H^2(P/Q;\mathbb{C}^*)$$

that is a left inverse of the map $[c] \mapsto [\mathscr{E}_c]$.

Assume $\mathscr{E}$ is an invariant dual 2-cocycle on G_q. Then it acts by multiplication by a nonzero scalar on every isotypic component of $V_\mu \otimes V_\eta$ of multiplicity one. In particular, consider the isotypic component corresponding to the weight $\mu+\eta$. It is the image of the map $T_{\mu,\eta}\colon V_{\mu+\eta} \to V_\mu \otimes V_\eta$, $\xi_{\mu+\eta} \mapsto \xi_\mu \otimes \xi_\eta$, introduced in the proof of Lemma 3.3.4. Hence there exists $\varepsilon(\mu,\eta) \in \mathbb{C}^*$ such that

$$\mathscr{E} T_{\mu,\eta} = \varepsilon(\mu,\eta) T_{\mu,\eta}.$$

Lemma 3.4.3. — *The map $P_+ \times P_+ \to \mathbb{C}^*$, $(\mu,\eta) \mapsto \varepsilon(\mu,\eta)$, is a 2-cocycle on P_+, that is,*

$$\varepsilon(\mu,\eta)\varepsilon(\mu+\eta,\nu) = \varepsilon(\eta,\nu)\varepsilon(\mu,\eta+\nu).$$

Furthermore, the cohomology class $[\varepsilon]$ of ε in $H^2(P_+;\mathbb{C}^*)$ depends only on the class of $\mathscr{E}$ in $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$.

Proof. — The assertion that ε is a cocycle follows from the identity

$$(T_{\mu,\eta} \otimes \iota) T_{\mu+\eta,\nu} = (\iota \otimes T_{\eta,\nu}) T_{\mu,\eta+\nu}$$

by applying the operator $(\mathscr{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathscr{E})$ to the left hand side and the same operator $(1 \otimes \mathscr{E})(\iota \otimes \hat{\Delta}_q)(\mathscr{E})$ to the right hand side.

Note that if $a \in \mathscr{U}(G_q)$ is a central element acting on V_μ as multiplication by a scalar $a(\mu)$, then the action of $(a \otimes a)\hat{\Delta}_q(a)^{-1}$ on the image of $T_{\mu,\eta}$ is given by multiplication by $a(\mu)a(\eta)a(\mu+\eta)^{-1}$. This shows that the cohomology class of ε depends only on that of $\mathscr{E}$. □

We claim that ε is cohomologous to a cocycle defined on P/Q. We need some preparation to prove this.

Fix $1 \le i \le r$. For weights $\mu,\eta \in P_+$ with $\mu(i), \eta(i) \ge 1$ the isotypic component of $V_\mu \otimes V_\eta$ corresponding to the weight $\mu+\eta-\alpha_i$ is nonzero. It is the image of the morphism

$$\tau_{i;\mu,\eta}\colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta$$

such that

$$\xi_{\mu+\eta-\alpha_i} \mapsto [\mu(i)]_{q_i} \xi_\mu \otimes F_i \xi_\eta - q_i^{\mu(i)} [\eta(i)]_{q_i} F_i \xi_\mu \otimes \xi_\eta,$$

which we introduced in the proof of Theorem 3.3.7. Hence there exists $\varepsilon_i(\mu,\eta) \in \mathbb{C}^*$ such that

$$\mathscr{E} \tau_{i;\mu,\eta} = \varepsilon_i(\mu,\eta) \tau_{i;\mu,\eta}.$$

Lemma 3.4.4. — *For any $\mu, \eta, \nu \in P_+$ with $\mu(i), \eta(i), \nu(i) \geq 1$ we have*

$$\begin{aligned}\varepsilon_i(\mu+\eta,\nu)\varepsilon(\mu,\eta) &= \varepsilon_i(\mu,\eta)\varepsilon(\mu+\eta-\alpha_i,\nu)\\ &= \varepsilon_i(\mu,\eta+\nu)\varepsilon(\eta,\nu) = \varepsilon_i(\eta,\nu)\varepsilon(\mu,\eta+\nu-\alpha_i).\end{aligned}$$

Proof. — Consider the module $V_\mu \otimes V_\eta \otimes V_\nu$. We have

$$\begin{aligned}V_\mu \otimes V_\eta \otimes V_\nu &\cong (V_{\mu+\eta} \oplus V_{\mu+\eta-\alpha_i} \oplus \dots) \otimes V_\nu\\ &\cong (V_{\mu+\eta+\nu} \oplus V_{\mu+\eta+\nu-\alpha_i} \oplus \dots) \oplus (V_{\mu+\eta+\nu-\alpha_i} \oplus \dots) \oplus \dots .\end{aligned}$$

We see that the isotypic component corresponding to $\mu+\eta+\nu-\alpha_i$ has multiplicity two, and it is spanned by the images of $(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu}$ and $(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$. Similarly, if we first consider the decomposition of $V_\eta \otimes V_\nu$, we conclude that the same isotypic component is spanned by the images of $(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu}$ and $(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$. These four maps are related by the identities

$$\begin{aligned}[\eta(i)]_{q_i}(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu} - [\nu(i)]_{q_i}(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}&\\ = [\mu(i)+\eta(i)]_{q_i}(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}&,\end{aligned} \tag{3.4.1}$$

$$\begin{aligned}[\eta(i)]_{q_i}(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu} - [\mu(i)]_{q_i}(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}&\\ = [\eta(i)+\nu(i)]_{q_i}(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}&,\end{aligned} \tag{3.4.2}$$

as can be checked by applying both sides to the highest weight vector $\xi_{\mu+\eta+\nu-\alpha_i}$.

The morphisms $(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu}$, $(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$, $(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu}$ and $(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$ are eigenvectors of the operator of multiplication by

$$(\mathscr{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathscr{E}) = (1 \otimes \mathscr{E})(\iota \otimes \hat{\Delta}_q)(\mathscr{E})$$

on the left, with eigenvalues $\varepsilon(\mu,\eta)\varepsilon_i(\mu+\eta,\nu)$, $\varepsilon_i(\mu,\eta)\varepsilon(\mu+\eta-\alpha_i,\nu)$, $\varepsilon(\eta,\nu)\varepsilon_i(\mu,\eta+\nu)$ and $\varepsilon_i(\eta,\nu)\varepsilon(\mu,\eta+\nu-\alpha_i)$, respectively. At the same time these eigenvectors have the property that any two of them are linearly independent and any three of them are linearly dependent. This is possible only when all four eigenvalues coincide. □

We will also need to know that any symmetric cocycle on P_+ is a coboundary. This can be easily deduced from the corresponding well-known statement for abelian groups, but for future reference we will establish a more precise result.

Lemma 3.4.5. — *Let $(\mu,\eta) \mapsto c(\mu,\eta)$ be a symmetric $\mathbb{C}^*$-valued 2-cocycle on P_+. Then for any nonzero complex numbers $b_1, \dots, b_r$ there exists a unique map $P_+ \ni \mu \mapsto b(\mu) \in \mathbb{C}^*$ such that*

$$c(\mu,\eta) = b(\mu+\eta)b(\mu)^{-1}b(\eta)^{-1}, \;\; b(\omega_i) = b_i \;\; \textit{for } i = 1, \dots, r,$$

where $\omega_1, \dots, \omega_r$ are the fundamental weights.

Proof. — It is clear that the map b is unique, if it exists. To show the existence, for a weight $\mu \in P_+$, $\mu = k_1\omega_1 + \cdots + k_r\omega_r$, put $|\mu| = k_1 + \cdots + k_r$. Define $b(\mu)$ by induction on $|\mu|$ as follows. If $\mu - \omega_i$ is dominant for some i, then put

$$b(\mu) = c(\mu - \omega_i, \omega_i)b(\mu - \omega_i)b(\omega_i).$$

We have to check that $b(\mu)$ is well-defined. In other words, if $\mu = \nu + \omega_i + \omega_j$, then we must show that

$$c(\nu + \omega_j, \omega_i)b(\nu + \omega_j)b(\omega_i) = c(\nu + \omega_i, \omega_j)b(\nu + \omega_i)b(\omega_j).$$

Using the cocycle identities $c(\nu + \omega_j, \omega_i)c(\nu, \omega_j) = c(\nu, \omega_j + \omega_i)c(\omega_j, \omega_i)$ and

$$c(\nu + \omega_i, \omega_j)c(\nu, \omega_i) = c(\nu, \omega_i + \omega_j)c(\omega_i, \omega_j),$$

as well as that $c(\omega_i, \omega_j) = c(\omega_j, \omega_i)$ by assumption, we equivalently have to check that

$$c(\nu, \omega_i)b(\nu + \omega_j)b(\omega_i) = c(\nu, \omega_j)b(\nu + \omega_i)b(\omega_j).$$

Since $c(\nu, \omega_i) = b(\nu + \omega_i)b(\nu)^{-1}b(\omega_i)^{-1}$ and $c(\nu, \omega_j) = b(\nu + \omega_j)b(\nu)^{-1}b(\omega_j)^{-1}$ by the inductive assumption, this identity indeed holds.

Therefore we have constructed a map b such that $b(0) = c(0,0)$, $b(\omega_i) = b_i$ and $c(\mu, \omega_i) = b(\mu + \omega_i)b(\mu)^{-1}b(\omega_i)^{-1}$ for $i = 1, \dots, r$ and $\mu \in P_+$. By induction on $|\eta|$ one can easily check that the identity $c(\mu, \eta) = b(\mu + \eta)b(\mu)^{-1}b(\eta)^{-1}$ holds for all $\mu, \eta \in P_+$. □

Given a cocycle on P/Q, we can consider it as a cocycle on P and then get a cocycle on P_+ by restriction. Thus we have a homomorphism $H^2(P/Q; \mathbb{C}^*) \to H^2(P_+; \mathbb{C}^*)$. It is injective, since the quotient map $P_+ \to P/Q$ is surjective and a cocycle on P/Q is a coboundary if it is symmetric.

Lemma 3.4.6. — *For every invariant 2-cocycle $\mathscr{E}$ on $\hat{G}_q$ the cohomology class of ε in $H^2(P_+; \mathbb{C}^*)$ is contained in the image of $H^2(P/Q; \mathbb{C}^*)$.*

Proof. — Put $b(\mu, \eta) = \varepsilon(\mu, \eta)\varepsilon(\eta, \mu)^{-1}$. The same computation as in the group case shows that b is a skew-symmetric bi-quasi-character, so it is a $\mathbb{C}^*$-valued function such that

$$b(\mu + \eta, \nu) = b(\mu, \nu)b(\eta, \nu), \quad b(\mu, \eta + \nu) = b(\mu, \eta)b(\mu, \nu), \quad b(\mu, \mu) = 1.$$

It extends uniquely to a skew-symmetric bi-quasi-character on P. To prove the lemma it suffices to show that the root lattice Q is contained in the kernel of this extension. Indeed, since $H^2(P/Q; \mathbb{C}^*)$ is isomorphic to the group of skew-symmetric bi-characters on P/Q, it then follows that there exists a cocycle c on P/Q such that the cocycle εc^{-1} on P_+ is symmetric. Then by Lemma 3.4.5 the cocycle εc^{-1} is a coboundary, so ε and the restriction of c to P_+ are cohomologous.

By Lemma 3.4.4, for any $1 \le i \le r$ we have

$$\varepsilon_i(\mu,\eta)\varepsilon(\mu+\eta-\alpha_i,\nu) = \varepsilon_i(\eta,\nu)\varepsilon(\mu,\eta+\nu-\alpha_i).$$

Applying this to $\eta=\nu=\mu$ we get

$$b(2\mu-\alpha_i,\mu)=1.$$

Since b is skew-symmetric, this gives $b(\alpha_i,\mu)=1$. This identity holds for all $\mu\in P_+$ with $\mu(i)\ge 1$. Since every element in P can be written as a difference of two such elements μ, it follows that α_i is contained in the kernel of b. □

Hence the map $\mathscr{E}\mapsto\varepsilon$ induces a homomorphism $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)\to H^2(P/Q;\mathbb{C}^*)$. Clearly, it is a left inverse of the homomorphism $H^2(P/Q;\mathbb{C}^*)\to H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$, $[c]\mapsto[\mathscr{E}_c]$, constructed earlier. Thus it remains to prove that the homomorphism $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)\to H^2(P/Q;\mathbb{C}^*)$ is injective.

Assume that $\mathscr{E}$ is an invariant 2-cocycle such that the cocycle ε on P_+ is a coboundary. We have to show that $\mathscr{E}$ is the coboundary of a central element in $\mathscr{U}(G_q)$.

By assumption there exist $c(\mu)\in\mathbb{C}^*$ such that

$$\varepsilon(\mu,\eta)=c(\mu+\eta)c(\mu)^{-1}c(\eta)^{-1}.$$

The numbers $c(\mu)$, $\mu\in P_+$, define an invertible element c in the center of $\mathscr{U}(G_q)$. Then replacing $\mathscr{E}$ by $(c^{-1}\otimes c^{-1})\mathscr{E}\hat{\Delta}_q(c)$ we get a new invariant 2-cocycle that is cohomologous to $\mathscr{E}$ and is such that the corresponding 2-cocycle on P_+ is trivial. In other words, without loss of generality we may assume that

$$\varepsilon(\mu,\eta)=1 \ \text{ for all } \ \mu,\eta\in P_+. \tag{3.4.3}$$

Note that this in particular implies that $\mathscr{E}$ is counital, since $(\hat{\varepsilon}_q\otimes\iota)(\mathscr{E})$ acts on V_μ as multiplication by $\varepsilon(0,\mu)$.

As before, let $\varepsilon_i(\mu,\eta)$ be such that $\mathscr{E}\tau_{i;\mu,\eta}=\varepsilon_i(\mu,\eta)\tau_{i;\mu,\eta}$.

Lemma 3.4.7. — *Assume the cocycle $\mathscr{E}$ satisfies condition* (3.4.3)*. Then, for every $1\le i\le r$, the numbers $\varepsilon_i(\mu,\eta)$ do not depend on $\mu,\eta\in P_+$ such that $\mu(i),\eta(i)\ge 1$.*

Proof. — By Lemma 3.4.4, we have

$$\varepsilon_i(\mu,\eta)=\varepsilon_i(\eta,\nu)$$

for all $\mu,\eta,\nu\in P_+$ with $\mu(i),\eta(i),\nu(i)\ge1$. For arbitrary $\mu,\eta,\tilde{\mu},\tilde{\eta}$, applying this identity twice, we get $\varepsilon_i(\mu,\eta)=\varepsilon_i(\eta,\tilde{\mu})=\varepsilon_i(\tilde{\mu},\tilde{\eta})$. □

Define a homomorphism $\chi\colon Q\to\mathbb{C}^*$ by letting $\chi(\alpha_i)=\varepsilon_i(\mu,\eta)^{-1}$ for $\mu,\eta\in P_+$ with $\mu(i),\eta(i)\ge1$, $1\le i\le r$. Extend χ to a homomorphism $P\to\mathbb{C}^*$. The restriction of χ to P_+ defines a central element c of $\mathscr{U}(G_q)$ such that

$$(c\otimes c)\hat{\Delta}_q(c)^{-1}\tau_{i;\mu,\eta}=\chi(\mu)\chi(\eta)\chi(\mu+\eta-\alpha_i)^{-1}\tau_{i;\mu,\eta}=\chi(\alpha_i)\tau_{i;\mu,\eta}=\varepsilon_i(\mu,\eta)^{-1}\tau_{i;\mu,\eta}.$$

Thus replacing $\mathscr{E}$ by the cohomologous cocycle $(c\otimes c)\mathscr{E}\hat{\Delta}_q(c)^{-1}$ we get an invariant 2-cocycle, which we again denote by $\mathscr{E}$, such that

$$\varepsilon_i(\mu,\eta)=1 \ \text{ for all } \ 1\le i\le r \ \text{ and } \ \mu,\eta\in P_+ \ \text{ with } \ \mu(i),\eta(i)\ge 1. \tag{3.4.4}$$

Note that condition (3.4.3) for this new cocycle is still satisfied, since χ is a homomorphism on P_+.

Therefore to prove the injectivity of $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)\to H^2(P/Q;\mathbb{C}^*)$ it suffices to establish the following result.

Proposition 3.4.8. — *If $\mathscr{E}$ is an invariant 2-cocycle on $\hat{G}_q$ with properties* (3.4.3) *and* (3.4.4), *then $\mathscr{E}=1$.*

For $G=SU(2)$ the weight lattice P is identified with $\frac12\mathbb{Z}$ and the root lattice with $\mathbb{Z}$. For $s\in\frac12\mathbb{N}$, we have $V_{1/2}\otimes V_s\cong V_{s+1/2}\oplus V_{s-1/2}$. Therefore conditions (3.4.3) and (3.4.4) imply that $\mathscr{E}$ acts trivially on $V_{1/2}\otimes V_s$.

Now for $s,t\ge 1/2$ consider the morphism $T_{1/2,s}\otimes\iota\colon V_{s+1/2}\otimes V_t\to V_{1/2}\otimes V_s\otimes V_t$ and compute:

$$\begin{aligned}(T_{1/2,s}\otimes\iota)\mathscr{E}&=(\hat{\Delta}_q\otimes\iota)(\mathscr{E})(T_{1/2,s}\otimes\iota)\\&=(1\otimes\mathscr{E})(\iota\otimes\hat{\Delta}_q)(\mathscr{E})(\mathscr{E}^{-1}\otimes 1)(T_{1/2,s}\otimes\iota)\\&=(1\otimes\mathscr{E})(T_{1/2,s}\otimes\iota),\end{aligned}$$

since $\mathscr{E}$ acts trivially on $V_{1/2}\otimes V$ for any V. It follows that if $\mathscr{E}$ acts trivially on $V_s\otimes V_t$, it acts trivially on $V_{s+1/2}\otimes V_t$. Therefore an induction argument shows that $\mathscr{E}$ acts trivially on $V_s\otimes V_t$ for all s and t, so $\mathscr{E}=1$.

For general G one can similarly show that it suffices to check that $\mathscr{E}$ acts trivially on $V_{\omega_i}\otimes V_\mu$, but it is not clear whether it is possible to check this property directly using conditions (3.4.3) and (3.4.4). We will prove the proposition by showing first that $\mathscr{E}$ defines an automorphism of the Kazhdan-Lusztig comonoid M introduced in the previous section.

Let us start by making a few remarks. For $\lambda\in P$ let $\bar\lambda=-w_0\lambda$, where w_0 is the longest element of the Weyl group. It is known that if $\mu\in P_+$ then $-\bar\mu$ is the lowest weight of V_μ, so $\bar\mu$ is the highest weight of $\bar V_\mu$. For $1\le i\le r$ let $\bar i$ be such that $\alpha_{\bar i}=\bar\alpha_i$ and $\omega_{\bar i}=\bar\omega_i$. Recall that in Section 3.3 we introduced morphisms $\bar T_{\mu,\eta}$ and $\bar\tau_{i;\mu,\eta}$. The first is an isomorphism of $\bar V_{\mu+\eta}$ onto the isotypic component of $\bar V_\mu\otimes\bar V_\eta$ with lowest weight $-\mu-\eta$, that is, with highest weight $\bar\mu+\bar\eta$. The second is an isomorphism of $\bar V_{\mu+\eta-\alpha_i}$ onto the isotypic component with lowest weight $-\mu-\eta+\alpha_i$, hence with highest weight $\bar\mu+\bar\eta-\bar\alpha_i$. Therefore if we fix isomorphisms $\bar V_\nu\cong V_{\bar\nu}$, then $\bar T_{\mu,\eta}$ and $\bar\tau_{i;\mu,\eta}$ coincide with $T_{\bar\mu,\bar\eta}$ and $\tau_{\bar i;\bar\mu,\bar\eta}$ up to scalar factors. Hence properties (3.4.3) and (3.4.4) also imply that

$$\mathscr{E}\bar T_{\mu,\eta}=\bar T_{\mu,\eta} \ \text{ and } \ \mathscr{E}\bar\tau_{i;\mu,\eta}=\bar\tau_{i;\mu,\eta}.$$

Recall also that in Section 3.3 we introduced a morphism $S_\mu\colon \bar V_\mu\otimes V_\mu\to V_0=\mathbb{C}$ such that $S_\mu(\bar\xi_\mu\otimes\xi_\mu)=1$. Since $\mathscr{E}$ is invertible, the morphism $S_\mu\mathscr{E}\colon \bar V_\mu\otimes V_\mu\to\mathbb{C}$ is nonzero, hence it is a nonzero multiple of S_μ, so $S_\mu\mathscr{E}=\chi(\mu)S_\mu$ for some $\chi(\mu)\in\mathbb{C}^*$. Explicitly, $\chi(\mu)=S_\mu\mathscr{E}(\bar\xi_\mu\otimes\xi_\mu)$.

Finally, recall that in Lemma 3.3.4 we defined morphisms

$$\mathrm{tr}^\eta_{\mu,\lambda+\mu}=(\iota\otimes S_\eta\otimes\iota)(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu})\colon \bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\to\bar V_\mu\otimes V_{\lambda+\mu}.$$

Lemma 3.4.9. — *For all $\mu,\eta\in P_+$ and $\lambda\in P$ such that $\lambda+\mu\in P_+$ we have $\mathrm{tr}^\eta_{\mu,\lambda+\mu}\mathscr{E}=\chi(\eta)\mathscr{E}\,\mathrm{tr}^\eta_{\mu,\lambda+\mu}$.*

Proof. — Applying $\iota\otimes\iota\otimes\hat\Delta_q$ to the cocycle identity

$$(\mathscr{E}\otimes1)(\hat\Delta_q\otimes\iota)(\mathscr{E})=(1\otimes\mathscr{E})(\iota\otimes\hat\Delta_q)(\mathscr{E}),$$

we get

$$(\mathscr{E}\otimes1\otimes1)(\hat\Delta_q\otimes\hat\Delta_q)(\mathscr{E})=(1\otimes(\iota\otimes\hat\Delta_q)(\mathscr{E}))(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E}),$$

where $\hat\Delta_q^{(2)}=(\iota\otimes\hat\Delta_q)\hat\Delta_q$. Replacing $(\iota\otimes\hat\Delta_q)(\mathscr{E})$ by $(1\otimes\mathscr{E}^{-1})(\mathscr{E}\otimes1)(\hat\Delta_q\otimes\iota)(\mathscr{E})$ on the right hand side, we then get

$$(\mathscr{E}\otimes\mathscr{E})(\hat\Delta_q\otimes\hat\Delta_q)(\mathscr{E})=(1\otimes\mathscr{E}\otimes1)(1\otimes(\hat\Delta_q\otimes\iota)(\mathscr{E}))(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E}),$$

which can also be written as

$$(\hat\Delta_q\otimes\hat\Delta_q)(\mathscr{E})=(1\otimes\mathscr{E}\otimes1)(1\otimes(\hat\Delta_q\otimes\iota)(\mathscr{E}))(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E})(\mathscr{E}^{-1}\otimes\mathscr{E}^{-1}),$$

since $\mathscr{E}$ commutes with the image of $\hat\Delta_q$ by the invariance assumption.

We then compute:

$$\begin{aligned}\mathrm{tr}^\eta_{\mu,\lambda+\mu}\mathscr{E}&=(\iota\otimes S_\eta\otimes\iota)(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu})\mathscr{E}\\&=(\iota\otimes S_\eta\otimes\iota)(\hat\Delta_q\otimes\hat\Delta_q)(\mathscr{E})(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu})\\&=(\iota\otimes S_\eta\otimes\iota)(1\otimes\mathscr{E}\otimes1)(1\otimes(\hat\Delta_q\otimes\iota)(\mathscr{E}))\\&\qquad(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E})(\mathscr{E}^{-1}\otimes\mathscr{E}^{-1})(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}).\end{aligned}$$

By condition (3.4.3) the last expression equals

$$(\iota\otimes S_\eta\otimes\iota)(1\otimes\mathscr{E}\otimes1)(1\otimes(\hat\Delta_q\otimes\iota)(\mathscr{E}))(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E})(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}).$$

Since $S_\eta\mathscr{E}=\chi(\eta)S_\eta$, $S_\eta\hat\Delta_q(\omega)=\hat\varepsilon_q(\omega)S_\eta$ and $(\hat\varepsilon_q\otimes\iota)(\mathscr{E})=1$ by (3.4.3), this expression equals

$$\chi(\eta)(\iota\otimes S_\eta\otimes\iota)(\iota\otimes\hat\Delta_q^{(2)})(\mathscr{E})(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}),$$

and using again $S_\eta\hat\Delta_q(\omega)=\hat\varepsilon_q(\omega)S_\eta$ and $(\hat\varepsilon_q\otimes\iota)\hat\Delta_q=\iota$ we obtain

$$\chi(\eta)\mathscr{E}(\iota\otimes S_\eta\otimes\iota)(\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu})=\chi(\eta)\mathscr{E}\,\mathrm{tr}^\eta_{\mu,\lambda+\mu}.$$ □

In particular, using that $S_{\mu+\eta} = \mathrm{tr}^{\mu+\eta}_{0,0} = \mathrm{tr}^{\mu}_{0,0}\,\mathrm{tr}^{\eta}_{\mu,\mu}$ we get

$$\chi(\mu+\eta)S_{\mu+\eta} = S_{\mu+\eta}\mathscr{E} = S_\mu\,\mathrm{tr}^{\eta}_{\mu,\mu}\,\mathscr{E} = \chi(\eta)S_\mu\mathscr{E}\,\mathrm{tr}^{\eta}_{\mu,\mu}$$
$$= \chi(\eta)\chi(\mu)S_\mu\,\mathrm{tr}^{\eta}_{\mu,\mu} = \chi(\eta)\chi(\mu)S_{\mu+\eta}.$$

Thus the map $\chi\colon P_+ \to \mathbb{C}^*$ is a homomorphism, hence it extends to a homomorphism $P \to \mathbb{C}^*$, which we continue to denote by χ. This together with the above lemma implies that the morphisms

$$\chi(\mu)^{-1}\mathscr{E}\colon \bar V_\mu \otimes V_{\lambda+\mu} \to \bar V_\mu \otimes V_{\lambda+\mu}$$

are consistent with tr, hence define an automorphism $\mathscr{E}_0$ of the pro-object M.

Recall that according to Theorem 3.3.7 we have an action of $(U_q\mathfrak{g})^{\mathrm{op}}$ on M.

Lemma 3.4.10. — *The automorphism $\mathscr{E}_0$ commutes with the action of $(U_q\mathfrak{g})^{\mathrm{op}}$ on M.*

Proof. — We will assume that $q \neq 1$, the case $q = 1$ requires only minor changes. Let us show first that for all $1 \le i \le r$ we have

$$\tilde E_i\mathscr{E}_0 = \chi(\alpha_i)\mathscr{E}_0\tilde E_i,\quad \tilde F_i\mathscr{E}_0 = \mathscr{E}_0\tilde F_i \quad\text{and}\quad \tilde K_i\mathscr{E}_0 = \mathscr{E}_0\tilde K_i.$$

The morphism $\tilde E_i$ is defined using the morphisms $\Phi^{\eta}_{i;\mu+\alpha_i,\lambda+\mu}$ given by the composition

$$\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]^{-1}_{q_i}\bar\tau_{i;\mu+\alpha_i,\eta}\otimes T_{\eta,\lambda+\mu}} \bar V_{\mu+\alpha_i}\otimes \bar V_\eta\otimes V_\eta\otimes V_{\lambda+\mu} \xrightarrow{\iota\otimes S_\eta\otimes\iota} \bar V_{\mu+\alpha_i}\otimes V_{\lambda+\mu}.$$

The same proof as that of Lemma 3.4.9 shows that

$$\Phi^{\eta}_{i;\mu+\alpha_i,\lambda+\mu}\mathscr{E} = \chi(\eta)\mathscr{E}\Phi^{\eta}_{i;\mu+\alpha_i,\lambda+\mu}.$$

The only difference is that $\bar T_{\mu,\eta}$ in that lemma gets replaced by $\bar\tau_{i;\mu+\alpha_i,\eta}$ and then instead of condition (3.4.3) one uses condition (3.4.4). Dividing both sides of the above identity by $\chi(\mu+\eta)$, we get $\tilde E_i\mathscr{E}_0 = \chi(\alpha_i)\mathscr{E}_0\tilde E_i$.

Similarly, $\tilde F_i$ is defined using the morphisms $\Psi^{\eta}_{i;\mu,\lambda+\alpha_i+\mu}$ given by the composition

$$\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]^{-1}_{q_i}\bar T_{\mu,\eta}\otimes\tau_{i;\eta,\lambda+\alpha_i+\mu}} \bar V_{\mu}\otimes \bar V_\eta\otimes V_\eta\otimes V_{\lambda+\alpha_i+\mu} \xrightarrow{\iota\otimes S_\eta\otimes\iota} \bar V_{\mu}\otimes V_{\lambda+\alpha_i+\mu}.$$

It follows that

$$\Psi^{\eta}_{i;\mu,\lambda+\alpha_i+\mu}\mathscr{E} = \chi(\eta)\mathscr{E}\Psi^{\eta}_{i;\mu,\lambda+\alpha_i+\mu},$$

and dividing both sides by $\chi(\mu+\eta)$ we get $\tilde F_i\mathscr{E}_0 = \mathscr{E}_0\tilde F_i$.

The commutation with $\tilde K_i$ is obvious.

Since $\tilde E_i\tilde F_i - \tilde F_i\tilde E_i$ coincides with $\tilde K_i - \tilde K_i^{-1}$ up to a scalar factor, we conclude that

$$(\chi(\alpha_i)-1)\mathscr{E}_0(\tilde K_i - \tilde K_i^{-1}) = 0.$$

Since $\mathscr{E}_0$ is invertible, this is possible only when $\chi(\alpha_i) = 1$. Hence χ is trivial on Q and $\mathscr{E}_0$ commutes with the action of $(U_q\mathfrak{g})^{\mathrm{op}}$. $\square$

Proof of Proposition 3.4.8. — Consider the functor $F = \mathrm{Hom}(M, \cdot)$ from $\mathscr{C}_q(\mathfrak{g})$ into the category of finite dimensional admissible $U_q\mathfrak{g}$-modules. The automorphism $\mathscr{E}_0$ of M defines an automorphism of F that maps $f\colon M \to V$ into $f\mathscr{E}_0$. As we showed in Section 3.3, the functor F is isomorphic to the identity functor on $\mathscr{C}_q(\mathfrak{g})$. Hence $\mathscr{E}_0$ defines an automorphism of the latter functor. Since the algebra of endomorphisms of the identity functor can be identified with the center of $\mathscr{U}(G_q)$, the automorphism $\mathscr{E}_0$ therefore defines an invertible central element $c \in \mathscr{U}(G_q)$. By construction of M and the isomorphisms $\eta_V : F(V) \to V$, this means that for any morphism $f\colon \bar{V}_\mu \otimes V_{\lambda+\mu} \to V$ we have

$$\chi(\mu)^{-1} f(\mathscr{E}(\bar{\xi}_\mu \otimes \xi_{\lambda+\mu})) = cf(\bar{\xi}_\mu \otimes \xi_{\lambda+\mu}).$$

Since this is true for any f and V, and the vector $\bar{\xi}_\mu \otimes \xi_{\lambda+\mu}$ is cyclic, we get

$$\mathscr{E} = \chi(\mu)\hat{\Delta}_q(c) \ \text{ on } \ \bar{V}_\mu \otimes V_{\lambda+\mu}.$$

In other words, if we introduce a central element b that acts as multiplication by $\chi(\mu)$ on $\bar{V}_\mu \cong V_{\bar{\mu}}$, then

$$\mathscr{E} = (b \otimes 1)\hat{\Delta}_q(c).$$

Since $\mathscr{E}$ is counital, applying $\hat{\varepsilon}_q \otimes \iota$ and $\iota \otimes \hat{\varepsilon}_q$ to this identity we obtain

$$\hat{\varepsilon}_q(b)c = 1 = bc.$$

Hence b and c are scalar and $\mathscr{E} = 1$. This finishes the proof of Proposition 3.4.8 and thus also of Theorem 3.4.1. □

We now formulate an important corollary to Theorem 3.4.1. Recall that the category $\mathscr{C}_q(\mathfrak{g})$ is braided, with braiding $\sigma = \Sigma\mathscr{R}_q$.

Definition 3.4.11. — We say that an invariant dual 2-cocycle $\mathscr{E}$ on G_q is symmetric, if $\mathscr{E}$ commutes with σ, that is, $\mathscr{R}_q\mathscr{E} = \mathscr{E}_{21}\mathscr{R}_q$.

Corollary 3.4.12. — *If $\mathscr{E}$ is a symmetric invariant dual 2-cocycle on G_q, then $\mathscr{E} = (c \otimes c)\hat{\Delta}_q(c)^{-1}$ for an invertible central element $c \in \mathscr{U}(G_q)$. Furthermore, if $\mathscr{E}$ is unitary, then c can also be chosen to be unitary.*

Proof. — Since σ maps the isotypic component of $V_\mu \otimes V_\eta$ corresponding to the weight $\mu+\eta$ onto the isotypic component of $V_\eta \otimes V_\mu$ with the same weight, the identity $\mathscr{E}\sigma = \sigma\mathscr{E}$ implies that $\varepsilon(\eta,\mu) = \varepsilon(\mu,\eta)$. Therefore the cocycle ε on P_+ is symmetric. Hence it is a coboundary by Lemma 3.4.5, so the image of $\mathscr{E}$ in $H^2(P/Q; \mathbb{C}^*)$ is trivial. □

Finally, we remark that for $q = 1$ the above methods can be easily extended to all connected compact Lie groups. Using that any compact group is a projective limit of Lie groups, one then can prove the following.

Theorem 3.4.13. — *For any compact connected group G we have a canonical isomorphism*

$$H^2_G(\hat{G};\mathbb{T}) \cong H^2(\widehat{Z(G)};\mathbb{T}).$$

If G is in addition separable, then we also have a canonical isomorphism

$$H^2_G(\hat{G};\mathbb{C}^*) \cong H^2(\widehat{Z(G)};\mathbb{C}^*).$$

References. — [**38**], [**70**], [**71**].

CHAPTER 4

DRINFELD TWISTS

In this chapter we give a precise relation between the tensor categories of representations of a simply connected semisimple compact Lie group G and its q-deformation G_q. It turns out that these categories become equivalent once we replace the trivial associativity morphisms in $\operatorname{Rep} G$ by certain morphisms defined by monodromy of a remarkable system of differential equations. We then discuss operator algebraic implications of this result, as well as its most famous consequence, the Drinfeld-Kohno theorem.

4.1. DRINFELD CATEGORY

Throughout the whole chapter we denote by G a simply connected semisimple compact Lie group. As in Section 2.4, fix a nondegenerate symmetric ad-invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$ such that its restriction to the real Lie algebra of G is negative definite. Let $t=\sum_i x_i\otimes x^i\in\mathfrak{g}\otimes\mathfrak{g}$ be the element defined by this form, so $\{x_i\}_i$ is a basis in $\mathfrak{g}$ and $\{x^i\}_i$ is the dual basis. Explicitly, we can write

$$t=\sum_{i,j}(B^{-1})_{ij}H_i\otimes H_j+\sum_{\alpha\in\Delta_+}d_\alpha(F_\alpha\otimes E_\alpha+E_\alpha\otimes F_\alpha),$$

where B is the matrix $((\alpha_i^\vee,\alpha_j^\vee))_{i,j}=(d_j^{-1}a_{ij})_{i,j}$. By the ad-invariance of our fixed form, the element t is invariant, that is, it commutes with all elements of the form $\hat\Delta(\omega)$, $\omega\in U\mathfrak{g}$. Note also that since $\hat\Delta(X)=X\otimes 1+1\otimes X$ for $X\in\mathfrak{g}$, we have

$$(\hat\Delta\otimes\iota)(t)=t_{13}+t_{23},\quad(\iota\otimes\hat\Delta)(t)=t_{12}+t_{13}. \tag{4.1.1}$$

Fix a number $\hbar\in\mathbb{C}$. Let $V_1,\dots,V_n$ be finite dimensional $\mathfrak{g}$-modules. Denote by Y_n the set of points $(z_1,\dots,z_n)\in\mathbb{C}^n$ such that $z_i\ne z_j$ for $i\ne j$. The system of Knizhnik-Zamolodchikov equations in n variables is the system of differential equations

$$\frac{\partial v}{\partial z_i}=\hbar\sum_{j\ne i}\frac{t_{ij}}{z_i-z_j}v,\quad i=1,\dots,n,$$

where $v\colon Y_n \to V_1 \otimes \cdots \otimes V_n$. We denote it by KZ_n.

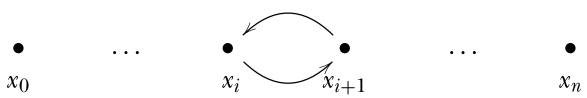
This system is consistent in the sense that the differential operators

$$\nabla_i = \frac{\partial}{\partial z_i} - \hbar \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j}$$

commute with each other, or in other words, they define a flat holomorphic connection on the trivial vector bundle over Y_n with fiber $V_1 \otimes \cdots \otimes V_n$. This is not difficult to check using that t is symmetric and that $[t_{ij} + t_{jk}, t_{ik}] = 0$, which follows from (4.1.1) and the invariance of t.

The consistency of the KZ_n equations implies that locally for each $z^0 \in Y_n$ and $v_0 \in V_1 \otimes \cdots \otimes V_n$ there exists a unique holomorphic solution v with $v(z^0) = v_0$. If $\gamma\colon [0,1] \to Y_n$ is a path starting at $\gamma(0) = z^0$, then this solution can be analytically continued along γ. The map $v_0 \mapsto v(\gamma(1))$ defines a linear isomorphism M_γ of $V_1 \otimes \cdots \otimes V_n$ onto itself. The monodromy operator M_γ depends only on the homotopy class of γ. In particular, for each base point $z^0 \in Y_n$ we get a representation of the fundamental group $\pi_1(Y_n; z^0)$ on $V_1 \otimes \cdots \otimes V_n$ by monodromy operators. This fundamental group is isomorphic to the pure braid group PB_n, which is the kernel of the canonical homomorphism from the braid group B_n into the symmetric group S_n. Recall that B_n is generated by elements $g_1, \dots, g_{n-1}$ satisfying the braid relations $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| > 1$. The subgroup PB_n is generated by the elements $g_i \dots g_{j-1}\, g_j^2\, g_{j-1}^{-1} \dots g_i^{-1}$, $1 \le i < j \le n-1$. If $V_1 = \cdots = V_n$, then the representation of PB_n by monodromy operators extends to the whole braid group. Namely, the action of S_n on Y_n and $V_1 \otimes \cdots \otimes V_n$ allows us to define a bundle over Y_n/S_n with fiber $V_1 \otimes \cdots \otimes V_n$, so we get a representation of $\pi_1(Y_n/S_n) \cong B_n$ on $V_1 \otimes \cdots \otimes V_n$ by monodromy operators.

More explicitly, choose a point $z^0 = (x_1^0, \dots, x_n^0)$ with real coordinates such that $x_i^0 < x_{i+1}^0$. Consider a path σ_i of the form



from z^0 to the point obtained by flipping the coordinates x_i^0 and x_{i+1}^0. Then, assuming that $V_1 = \cdots = V_n$, we get a representation of B_n on $V_1 \otimes \cdots \otimes V_n$ defined by

$$g_i \mapsto \Sigma_{i,i+1} M_{\sigma_i},$$

where $\Sigma_{i,i+1}$ is the flip on $V_i \otimes V_{i+1}$.

For $n = 2$ this representation is simply $g_1 \mapsto \Sigma e^{\pi i \hbar t}$. Consider the first nontrivial case, $n = 3$.

We look for solutions of KZ_3 of the form

$$v(z_1, z_2, z_3) = (z_3 - z_1)^{\hbar(t_{12}+t_{23}+t_{13})} w\left(\frac{z_2 - z_1}{z_3 - z_1}\right).$$

Then w must satisfy

$$w' = \hbar\left(\frac{t_{12}}{z} + \frac{t_{23}}{z-1}\right) w.$$

More generally, assume A and B are operators on a finite dimensional space V and consider solutions of the equation

$$w'(z) = \left(\frac{A}{z} + \frac{B}{z-1}\right) w(z). \tag{4.1.2}$$

We need the following standard result on differential equations with singularities, see e.g., [**68**, Proposition 3.3].

Proposition 4.1.1. — *Let $A_1, \dots, A_m \colon \mathbb{D}^m \to B(V)$, where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, be analytic functions such that the operators $z_i \frac{\partial}{\partial z_i} - A_i(z)$ pairwise commute. Assume that for every i the operator $A_i(0)$ has no eigenvalues that differ by a nonzero integer. Then the system of equations*

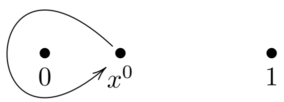
$$x_i \frac{\partial G}{\partial x_i} = A_i(x) G(x), \;\; 1 \le i \le m,$$

has a unique solution $G(x) \in B(V)$ on $(0,1)^m$ such that $G(x) x_1^{-A_1(0)} \dots x_m^{-A_m(0)}$ extends to an analytic function on $\mathbb{D}^m$ with value 1 at $x = 0$.

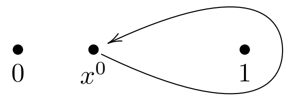
Note that by assumption the operators $A_i(0)$ pairwise commute. Observe also that since $G(0) = 1$ is invertible, the operator $G(z)$ is invertible for every $z \in \mathbb{D}^m$.

Apply the above proposition to Equation (4.1.2). Assume that neither A nor B has eigenvalues that differ by a nonzero integer. Then there is a unique $B(V)$-valued solution $G_0(x)$ of (4.1.2) on the interval $(0,1)$ such that $G_0(x)x^{-A}$ extends to a holomorphic function on $\mathbb{D}$ with value 1 at 0. Using the change of variables $z \mapsto 1 - z$ we similarly conclude that there is a unique $B(V)$-valued solution G_1 of (4.1.2) on $(0,1)$ such that $G_1(1-x)x^{-B}$ extends to a holomorphic function on $\mathbb{D}$ with value 1 at 0.

Fix $x^0 \in (0,1)$. If $w_0 \in V$, then $G_0(x)w_0$ is a solution of (4.1.2) with initial value $G_0(x^0)w_0$. If we continue it analytically along a loop γ_0 of the form



then at the end point we get $G_0(x^0)e^{2\pi i A}w_0$. Thus the monodromy operator defined by γ_0 is $G_0(x^0)e^{2\pi i A}G_0(x^0)^{-1}$. Similarly, the monodromy operator defined by a loop γ_1 of the form



is $G_1(x^0)e^{2\pi iB}G_1(x^0)^{-1}$. The fundamental group of $\mathbb{C}\setminus\{0,1\}$ with the base point x^0 is freely generated by the classes $[\gamma_0]$ and $[\gamma_1]$ of γ_0 and γ_1. Therefore the monodromy representation defined by Equation (4.1.2) with the base point x^0 is

$$[\gamma_0]\mapsto G_0(x^0)e^{2\pi iA}G_0(x^0)^{-1},\ \ [\gamma_1]\mapsto G_1(x^0)e^{2\pi iB}G_1(x^0)^{-1}.$$

This motivates introduction of the operator $\Phi(A,B) = G_1(x)^{-1}G_0(x)$. It does not depend on x, since a solution of (4.1.2) is determined by its initial value. We then see that the above representation of $\pi_1(\mathbb{C}\setminus\{0,1\})$ is equivalent to the representation

$$[\gamma_0]\mapsto e^{2\pi iA},\ \ [\gamma_1]\mapsto \Phi(A,B)^{-1}e^{2\pi iB}\Phi(A,B).$$

Note that we get something interesting only when A and B do not commute, since if they do commute, then $G_0(x) = G_1(x) = x^A(1-x)^B$ and $\Phi(A,B) = 1$.

The operator $\Phi(A,B)$ should be thought of as a normalized monodromy of (4.1.2) along the straight line from 0 to 1. Specifically, for $a\in(0,1)$ let G_a be the unique $B(V)$-valued solution of (4.1.2) on $(0,1)$ such that $G_a(a) = 1$. By definition the operator $G_a(b)$ is the monodromy from a to b.

Lemma 4.1.2. — *Assume V is a Hilbert space and the operators A and B are skew-adjoint. Then*

$$\Phi(A,B) = \lim_{a\downarrow 0} a^{-B}G_a(1-a)a^A.$$

In particular, the operator $\Phi(A,B)$ is unitary.

Proof. — Since a solution of (4.1.2) is determined by its initial value, we have $G_a(x) = G_0(x)G_0(a)^{-1}$. Hence

$$\begin{aligned} a^{-B}G_a(1-a)a^A &= a^{-B}G_0(1-a)G_0(a)^{-1}a^A\\ &= a^{-B}G_1(1-a)\Phi(A,B)G_0(a)^{-1}a^A. \end{aligned}$$

Note next that since a^B is unitary for all $a\in(0,1)$, we have

$$\lim_{a\downarrow 0} a^{-B}G_1(1-a) = \lim_{a\downarrow 0} a^{-B}(G_1(1-a)a^{-B})a^B = \lim_{a\downarrow 0} a^{-B}a^B = 1.$$

Similarly $a^{-A}G_0(a) \to 1$. This proves the first statement in the formulation of the lemma. For the second one observe that since G_a is an integral curve of a time-dependent vector field on the unitary group, the operator $G_a(x)$ is unitary for all $x\in(0,1)$. ☐

Note that under the assumptions of the lemma the operators $G_0(x)$ and $G_1(x)$ are unitary as well, since, for example,

$$G_0(x) = G_a(x)G_0(a) = G_a(x)(G_0(a)a^{-A})a^A,$$

and the last operator is close to the unitary operator $G_a(x)a^A$ if a is sufficiently small.

For arbitrary A and B the operator $\Phi(\hbar A, \hbar B)$ is well-defined at least for all $\hbar \in \mathbb{C}$ outside the discrete set Λ of numbers $n/(\lambda - \mu)$, where n is a nonzero integer and λ and μ are different eigenvalues either of A or of B. It can happen that $\Phi(\hbar A, \hbar B)$ is well-defined on a larger set. For example, if V decomposes into a direct sum of two A-and B-invariant subspaces V_1 and V_2, then it suffices to require $\hbar \neq n/(\lambda-\mu)$, where λ and μ are different eigenvalues of $A|_{V_i}$ or of $B|_{V_i}$.

Using the analyticity of solutions of differential equations with analytic coefficients it is easy to show that $\Phi(\hbar A, \hbar B)$ depends analytically on $\hbar \in \mathbb{C} \setminus \Lambda$. It can further be shown that the first terms of the Taylor series of $\Phi(\hbar A, \hbar B)$ at zero have the form

$$\Phi(\hbar A, \hbar B) = 1 - \hbar^2\zeta(2)[A,B] - \hbar^3\zeta(3)([A,[A,B]] + [B,[A,B]]) + \dots,$$

where ζ is the Riemann zeta function.

Returning to the KZ equations, note that the operator t is self-adjoint, so its spectrum consists of real numbers. Hence the operator $\Phi(\hbar t_{12}, \hbar t_{23})$ on $V_1 \otimes V_2 \otimes V_3$ is well-defined for all $\hbar$ outside a discrete subset of $\mathbb{R}^*$. Since this is true for all V_1, V_2, V_3, we get a well-defined element of $\mathscr{U}(G^3)$ for all $\hbar$ outside a countable subset of $\mathbb{R}^*$. This element is invariant, since t_{12} and t_{23} are invariant. If our fixed invariant form is standardly normalized, then the spectrum of t consists of rational numbers, so $\Phi(\hbar t_{12}, \hbar t_{23}) \in \mathscr{U}(G^3)$ is well-defined for all $\hbar \in \mathbb{C} \setminus \mathbb{Q}^*$. If $\hbar \in i\mathbb{R}$, then $\Phi(\hbar t_{12}, \hbar t_{23})$ is unitary.

The elements $\Phi(\hbar t_{12}, \hbar t_{23})$, when they are defined, completely describe the monodromy of the KZ_3 equations. Namely, assume V is a finite dimensional $\mathfrak{g}$-module and consider $V^{\otimes 3}$-valued solutions of KZ_3. There exist two $B(V^{\otimes 3})$-valued solutions W_0 and W_1 of KZ_3 on $\{x_1 < x_2 < x_3\}$ of the form

$$W_i(x_1, x_2, x_3) = (x_3 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} G_i\left(\frac{x_2 - x_1}{x_3 - x_1}\right), \quad i = 0, 1,$$

such that the functions $G_0(x)x^{-\hbar t_{12}}$ and $G_1(1-x)x^{-\hbar t_{23}}$ extend to analytic functions on the unit disc with value 1 at $x = 0$. From our discussion of monodromy of (4.1.2) it is then not difficult to see that if we fix a base point $z^0 = (x^0_1, x^0_2, x^0_3)$ with $x^0_1 < x^0_2 < x^0_3$, then the corresponding representation of the braid group is given by

$$g_1 \mapsto W_0(z^0)\Sigma_{12}e^{\pi i\hbar t_{12}}W_0(z^0)^{-1}, \quad g_2 \mapsto W_1(z^0)\Sigma_{23}e^{\pi i\hbar t_{23}}W_1(z^0)^{-1}.$$

Conjugating by $W_0(z^0)^{-1}$ we see that this representation is equivalent to

$$g_1 \mapsto \Sigma_{12}e^{\pi i\hbar t_{12}}, \quad g_2 \mapsto \Phi(\hbar t_{12}, \hbar t_{23})^{-1}\Sigma_{23}e^{\pi i\hbar t_{23}}\Phi(\hbar t_{12}, \hbar t_{23}).$$

This representation can be thought of as one corresponding to the base point at infinity in the asymptotic zone $x_2 - x_1 \ll x_3 - x_1$.

The elements $\Phi(\hbar t_{12}, \hbar t_{23})$ have the following remarkable property.

Theorem 4.1.3. — *For $\hbar \in \mathbb{C}$ outside a countable subset of $\mathbb{R}^*$, the element $\Phi(\hbar t_{12}, \hbar t_{23}) \in \mathscr{U}(G^3)$ is a counital invariant dual 3-cocycle on G.*

Proof. — We will only sketch a proof of this theorem, referring the reader to the original paper of Drinfeld [**28**] or to [**68**] for (slightly) more details.

Recall from our discussion in Section 3.1 that the cocycle identity means that $\Phi(\hbar t_{12}, \hbar t_{23})$ considered as a map $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ satisfies the pentagon relation. Consider five solutions of KZ_4 on $\{x_1 < x_2 < x_3 < x_4\}$ corresponding to the vertices of the pentagon diagram normalized in the asymptotic zones defined by the following rule: if V_i and V_j are between parenthesis and V_k is outside, then $|x_i - x_j| \ll |x_k - x_i|$. Specifically, we consider solutions of the form

$$W_i = (x_4 - x_1)^{\hbar T} F_i(u, v), \quad 1 \le i \le 5,$$

where $T = t_{12} + t_{13} + t_{14} + t_{23} + t_{24} + t_{34}$ and u and v are defined as follows:

$$\begin{aligned}
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 &: \ u = \frac{x_2 - x_1}{x_3 - x_1}, \ \ v = \frac{x_3 - x_1}{x_4 - x_1}, \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 &: \ u = \frac{x_3 - x_2}{x_3 - x_1}, \ \ v = \frac{x_3 - x_1}{x_4 - x_1}, \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) &: \ u = \frac{x_3 - x_2}{x_4 - x_2}, \ \ v = \frac{x_4 - x_2}{x_4 - x_1}, \\
V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) &: \ u = \frac{x_4 - x_3}{x_4 - x_2}, \ \ v = \frac{x_4 - x_2}{x_4 - x_1}, \\
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) &: \ u = \frac{x_2 - x_1}{x_4 - x_1}, \ \ v = \frac{x_4 - x_3}{x_4 - x_1}.
\end{aligned}$$

The functions F_i are required to be solutions of appropriate equations such that their behavior near zero is as described in Proposition 4.1.1. For example, F_1 is the unique solution of

$$\left\{\begin{aligned}
u\frac{\partial F_1}{\partial u} &= \hbar\left(t_{12} + \frac{u}{u-1}t_{23} + \frac{uv}{uv-1}t_{24}\right)F_1 \\
v\frac{\partial F_1}{\partial v} &= \hbar\left(t_{12} + t_{13} + t_{23} + \frac{uv}{uv-1}t_{24} + \frac{v}{v-1}t_{34}\right)F_1
\end{aligned}\right. \tag{4.1.3}$$

such that the function $F_1(u,v)u^{-\hbar t_{12}}v^{-\hbar(t_{12}+t_{13}+t_{23})}$ extends to an analytic function in a neighborhood of zero with value 1 at $u = v = 0$. Therefore in the asymptotic zone $x_2 - x_1 \ll x_3 - x_1 \ll x_4 - x_1$ we have

$$W_1 \sim (x_2 - x_1)^{\hbar t_{12}} (x_3 - x_1)^{\hbar(t_{13}+t_{23})} (x_4 - x_1)^{\hbar(t_{14}+t_{24}+t_{34})}.$$

These five solutions are related by the identities

$$W_1 = W_2(\Phi \otimes 1), \quad W_2 = W_3(1 \otimes \hat{\Delta} \otimes 1)(\Phi), \quad W_3 = W_4(1 \otimes \Phi),$$

$$W_4 = W_5(1 \otimes 1 \otimes \hat{\Delta})(\Phi)^{-1}, \quad W_5 = W_1(\hat{\Delta} \otimes 1 \otimes 1)(\Phi)^{-1},$$

where $\Phi = \Phi(\hbar t_{12}, \hbar t_{23})$, from which the cocycle identity immediately follows. Let us for example check the first equality $W_1 = W_2(\Phi \otimes 1)$.

The function F_2 is the unique solution of

$$\left\{\begin{aligned} u\frac{\partial F_2}{\partial u} &= \hbar\left(t_{23} + \frac{u}{u-1}t_{12} + \frac{uv}{1-v+uv}t_{24}\right)F_2 \\ v\frac{\partial F_2}{\partial v} &= \hbar\left(t_{12} + t_{13} + t_{23} + \frac{uv-v}{1-v+uv}t_{24} + \frac{v}{1-v}t_{34}\right)F_2 \end{aligned}\right. \tag{4.1.4}$$

such that the function $F_2(u,v)u^{-\hbar t_{23}}v^{-\hbar(t_{12}+t_{13}+t_{23})}$ extends to an analytic function in a neighborhood of zero with value 1 at $u = v = 0$.

We have $W_1 = W_2\Theta$ for some Θ. Then

$$F_1(u,v) = F_2(1-u,v)\Theta.$$

For $u \in (0,1)$ fixed, the functions

$$v \mapsto F_1(u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}, \quad v \mapsto F_2(1-u,v)v^{-\hbar(t_{12}+t_{13}+t_{23})}$$

extend analytically to a neighbourhood of zero. Hence the function

$$v \mapsto v^{\hbar(t_{12}+t_{13}+t_{23})}\Theta v^{-\hbar(t_{12}+t_{13}+t_{23})}$$

also extends analytically. Since the operator $\hbar(t_{12} + t_{13} + t_{23})$ has no eigenvalues that differ by a nonzero integer, one can easily check that this is possible only when Θ commutes with $\hbar(t_{12} + t_{13} + t_{23})$. It follows that if we put

$$g_i(u) = F_i(u,v)v^{\hbar(t_{12}+t_{13}+t_{23})}\,|_{v=0},$$

then $g_1(u)u^{-\hbar t_{12}}$ and $g_2(u)u^{-\hbar t_{23}}$ are analytic in a neighbourhood of zero, with value 1 at $u = 0$, and

$$g_1(u) = g_2(1-u)\Theta.$$

The first equation in (4.1.3) implies that

$$u\frac{dg_1}{du} = \hbar\left(t_{12} + \frac{u}{u-1}t_{23}\right)g_1,$$

while the first equation in (4.1.4) implies that

$$u\frac{dg_2}{du} = \hbar\left(t_{23} + \frac{u}{u-1}t_{12}\right)g_2.$$

Hence $\Theta = \Phi(\hbar t_{12}, \hbar t_{23}) = \Phi(\hbar t_{12}, \hbar t_{23}) \otimes 1$ by definition of $\Phi(\hbar t_{12}, \hbar t_{23})$. □

The element $\Phi(\hbar t_{12}, \hbar t_{23})$ is called Drinfeld's KZ-associator. It defines new associativity morphisms on the monoidal category $\operatorname{Rep} G$. Denote this new monoidal category by $\mathscr{D}(\mathfrak{g}, \hbar)$. If $\hbar \in i\mathbb{R}$, then $\Phi(\hbar t_{12}, \hbar t_{23})$ is unitary, so $\mathscr{D}(\mathfrak{g}, \hbar)$ is a C*-tensor category.

Theorem 4.1.4. — *The operator $\sigma = \Sigma e^{\pi i \hbar t}$ defines a braiding on $\mathscr{D}(\mathfrak{g}, \hbar)$.*

Proof. — Let us sketch a proof of the commutativity of the first hexagon diagram, see [**27**] or [**68**] for more details. Consider six solutions W_i, $1 \le i \le 6$, of KZ$_3$ in the simply connected region $\Gamma = \{(z_1, z_2, z_3) \in Y_3 \mid \Im z_1 \le \Im z_2 \le \Im z_3\}$ normalized in the real asymptotic zones corresponding to the vertices of the hexagon diagram according to the following rule:

$$\begin{aligned}
(V_1 \otimes V_2) \otimes V_3 &: \ x_1 < x_2 < x_3, \ \ x_2 - x_1 \ll x_3 - x_1, \\
V_1 \otimes (V_2 \otimes V_3) &: \ x_1 < x_2 < x_3, \ \ x_3 - x_2 \ll x_3 - x_1, \\
V_1 \otimes (V_3 \otimes V_2) &: \ x_1 < x_3 < x_2, \ \ x_2 - x_3 \ll x_2 - x_1, \\
(V_1 \otimes V_3) \otimes V_2 &: \ x_1 < x_3 < x_2, \ \ x_3 - x_1 \ll x_2 - x_1, \\
(V_3 \otimes V_1) \otimes V_2 &: \ x_3 < x_1 < x_2, \ \ x_1 - x_3 \ll x_2 - x_3, \\
V_3 \otimes (V_1 \otimes V_2) &: \ x_3 < x_1 < x_2, \ \ x_2 - x_1 \ll x_2 - x_3.
\end{aligned}$$

For example, W_4 is the solution obtained by the analytic continuation to Γ of the solution of the form

$$(x_2 - x_1)^{\hbar(t_{12}+t_{23}+t_{13})} F_4\left(\frac{x_3 - x_1}{x_2 - x_1}\right)$$

on $\{x_1 < x_3 < x_2\}$ such that $F_4(x) x^{-\hbar t_{13}}$ extends to an analytic function in the unit disc with value 1 at $x = 0$.

We claim that these six solutions are related by the identities

$$W_1 = W_2 \Phi, \ \ W_2 = W_3 e^{\pi i \hbar t_{23}}, \ \ W_3 = W_4 \Phi_{132}^{-1},$$

$$W_4 = W_5 e^{\pi i \hbar t_{13}}, \ \ W_5 = W_6 \Phi_{312}, \ \ W_6 = W_1 e^{-\pi i \hbar (t_{13} + t_{23})},$$

where $\Phi = \Phi(\hbar t_{12}, \hbar t_{23})$. This gives

$$e^{-\pi i \hbar (t_{13}+t_{23})} \Phi_{312} e^{\pi i \hbar t_{13}} \Phi_{132}^{-1} e^{\pi i \hbar t_{23}} \Phi = 1,$$

which is exactly the first hexagon identity.

The above identities involving Φ are immediate by definition. Among the remaining three identities let us, for example, check that $W_4 = W_5 e^{\pi i \hbar t_{13}}$. In the connected component of

$$\{(z_1, z_2, z_3) \in \Gamma : |z_1 - z_3| < |z_1 - z_2|, \ |z_1 - z_3| < |z_2 - z_3|\}$$

that intersects $\{x_1 < x_3 < x_2\}$ and $\{x_3 < x_1 < x_2\}$ we asymptotically have

$$W_4 \sim (z_3 - z_1)^{\hbar t_{13}} (z_2 - z_1)^{\hbar(t_{12}+t_{23})}$$

and

$$W_5 \sim (z_1 - z_3)^{\hbar t_{13}} (z_2 - z_3)^{\hbar(t_{12}+t_{23})} \sim (z_1 - z_3)^{\hbar t_{13}} (z_2 - z_1)^{\hbar(t_{12}+t_{23})}.$$

Here $(z_3 - z_1)^{\hbar t_{13}}$ is obtained by analytic continuation from $\{x_1 < x_3 < x_2\}$ to Γ, while $(z_1 - z_3)^{\hbar t_{13}}$ is obtained by analytic continuation from $\{x_3 < x_1 < x_2\}$. Hence

$$(z_3 - z_1)^{\hbar t_{13}} = (z_1 - z_3)^{\hbar t_{13}} e^{\pi i \hbar t_{13}}.$$

Since $W_4 = W_5\Theta$ for some Θ, we can therefore conclude that $\Theta = e^{\pi i \hbar t_{13}}$.

The second hexagon identity follows from the first one thanks to the identity $\Phi^{-1} = \Phi_{321}$, which follows from $\Phi(A,B)^{-1} = \Phi(B,A)$, which in turn is easily checked by definition using the change of variables $z \mapsto 1 - z$ in (4.1.2). □

Therefore for all $\hbar \in \mathbb{C}$ outside a countable subset of $\mathbb{R}^*$ (more precisely, for all $\hbar \notin \cup_i \mathbb{Q}^* d_i^{-1}$, in particular, for all $\hbar \notin \mathbb{Q}^*$ when our fixed ad-invariant form is standardly normalized) we get a braided monoidal category $\mathscr{D}(\mathfrak{g},\hbar)$ of finite dimensional $\mathfrak{g}$-modules with associativity morphisms $\Phi(\hbar t_{12}, \hbar t_{23})$ and braiding $\Sigma e^{\pi i \hbar t}$. It is called the Drinfeld category. If $\hbar \in i\mathbb{R}$, then this is a braided C*-tensor category with self-adjoint braiding.

References. — [**2**], [**27**], [**28**], [**30**], [**48**], [**52**], [**68**].

4.2. EQUIVALENCE OF TENSOR CATEGORIES

The aim of this section is to sketch a proof of the following fundamental result, due to Drinfeld in the formal deformation setting and to Kazhdan and Lusztig in the analytic case.

Theorem 4.2.1. — *Assume $\hbar \in i\mathbb{R}$ and let $q = e^{\pi i \hbar}$. Then there exists a braided unitary monoidal equivalence between the Drinfeld category $\mathscr{D}(\mathfrak{g},\hbar)$ and the category $\mathscr{C}_q(\mathfrak{g})$ that maps an irreducible $\mathfrak{g}$-module with highest weight λ onto an irreducible $U_q\mathfrak{g}$-module with highest weight λ.*

Without the unitarity condition the result is true for all $\hbar \notin \cup_i \mathbb{Q}^* d_i^{-1}$, but since we prefer to consider only C*-tensor categories, we concentrate on the case $\hbar \in i\mathbb{R}$.

Let us explain the idea of the proof. We will use super- and subscript q for everything related to $\mathscr{C}_q(\mathfrak{g})$. We have to construct a fiber functor on $\mathscr{D}(\mathfrak{g},\hbar)$ whose endomorphism ring contains $U_q\mathfrak{g}$. For this it is enough to construct a comonoid in $\mathscr{D}(\mathfrak{g},\hbar)$, or in a larger category, and an action of $(U_q\mathfrak{g})^{\mathrm{op}}$ on it. In Section 3.3 we defined such a comonoid in pro-$\mathscr{C}_q(\mathfrak{g})$. Its construction used certain morphisms $V^q_\mu \to V^q_\eta \otimes V^q_\nu$, which were uniquely defined, up to scalar factors, by the fusion rules. If the theorem is true, there exists a similar comonoid in pro-$\mathscr{D}(\mathfrak{g},\hbar)$ defined using properly normalized morphisms $V_\mu \to V_\eta \otimes V_\nu$. Now the idea is to construct such a comonoid from scratch by repeating the construction of the Kazhdan-Lusztig comonoid and changing the normalization of the structure maps when needed. Why such changes will be necessary, is because the category $\mathscr{D}(\mathfrak{g},\hbar)$ is nonstrict, and therefore taking tensor products of our structure maps we have to take into account the associativity morphisms. The point is that independently of how complicated at some points this might be, the construction *must* work with suitable normalizations of the maps involved, if the theorem is true.

Turning to the proof, let us say that a statement holds for generic $\hbar$ if it holds for all except countably many $\hbar$. Our goal in this section is to show that $\mathscr{D}(\mathfrak{g},\hbar)$ and $\mathscr{C}_q(\mathfrak{g})$ are braided monoidally equivalent for generic $\hbar\in i\mathbb{R}$ (in fact, for generic $\hbar\in\mathbb{C}$). In the next section we will show that the equivalence can be chosen to be unitary and we will then extend the result to all $\hbar\in i\mathbb{R}$.

Let V_μ be an irreducible $\mathfrak{g}$-module with highest weight μ and a highest weight vector ξ_μ. As in Section 3.3, consider morphisms

$$T_{\mu,\eta}\colon V_{\mu+\eta}\to V_\mu\otimes V_\eta,\ \ \xi_{\mu+\eta}\mapsto\xi_\mu\otimes\xi_\eta,$$
$$\bar T_{\mu,\eta}\colon \bar V_{\mu+\eta}\to \bar V_\mu\otimes \bar V_\eta,\ \ \bar\xi_{\mu+\eta}\mapsto\bar\xi_\mu\otimes\bar\xi_\eta,$$
$$S_\mu\colon \bar V_\mu\otimes V_\mu\to V_0=\mathbb{C},\ \ \bar\xi_\mu\otimes\xi_\mu\mapsto 1.$$

We want to define morphisms

$$\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\to\bar V_\mu\otimes V_{\lambda+\mu}$$

as suitably normalized compositions

$$\begin{aligned}\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}&\xrightarrow{\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}}(\bar V_\mu\otimes\bar V_\eta)\otimes(V_\eta\otimes V_{\lambda+\mu})\\&\xrightarrow{\Phi_{1,2,34}}\bar V_\mu\otimes(\bar V_\eta\otimes(V_\eta\otimes V_{\lambda+\mu}))\\&\xrightarrow{\iota\otimes\Phi^{-1}}\bar V_\mu\otimes((\bar V_\eta\otimes V_\eta)\otimes V_{\lambda+\mu})\\&\xrightarrow{\iota\otimes S_\eta\otimes\iota}\bar V_\mu\otimes V_{\lambda+\mu},\end{aligned}$$

where $\Phi=\Phi(\hbar t_{12},\hbar t_{23})$. Note that instead of $(\iota\otimes\Phi^{-1})\Phi_{1,2,34}$ we could also use $(\Phi\otimes\iota)\Phi^{-1}_{12,3,4}$, but the result would be the same by the pentagon relation. The above morphisms should at least form a projective system.

Lemma 4.2.2. — *Denote by $g^\hbar(\mu,\eta)$ the image of $\bar\xi_{\mu+\eta}\otimes\xi_{\mu+\eta}$ under the composition*

$$\bar V_{\mu+\eta}\otimes V_{\mu+\eta}\xrightarrow{\bar T_{\mu,\eta}\otimes T_{\eta,\mu}}\bar V_\mu\otimes\bar V_\eta\otimes V_\eta\otimes V_\mu\xrightarrow{(\iota\otimes S_\eta\otimes\iota)B}\bar V_\mu\otimes V_\mu\xrightarrow{S_\mu}\mathbb{C},$$

where $B=(\iota\otimes\Phi^{-1})\Phi_{1,2,34}$. Then for generic $\hbar$ the map $(\mu,\eta)\mapsto g^\hbar(\mu,\eta)$ is a $\mathbb{C}^*$-valued symmetric 2-cocycle on P_+.

Proof. — As $g^\hbar(\mu,\eta)$ is analytic in $\hbar$ outside a discrete set and $g^0(\mu,\eta) = 1$, we conclude that $g^\hbar(\mu,\eta) \neq 0$ for generic $\hbar$.

To prove that $g^\hbar(\mu,\eta)$ is a symmetric cocycle we will use two identities. The first is

$$\Phi(T_{\mu,\eta}\otimes\iota)T_{\mu+\eta,\nu} = (\iota\otimes T_{\eta,\nu})T_{\mu,\eta+\nu}.$$

To show this it suffices to check how both sides act on the highest weight vector $\xi_{\mu+\eta+\nu}$. But then this identity is immediate, since $\xi_\mu\otimes\xi_\eta\otimes\xi_\nu$ is an eigenvector for t_{12} and t_{23} and hence Φ acts trivially on it. The second identity is

$$\sigma T_{\mu,\eta} = q^{(\mu,\eta)}T_{\eta,\mu},$$

which is again straightforward to check by looking at how both sides act on $\xi_{\mu+\eta}$; recall that $\sigma = \Sigma q^t = \Sigma e^{\pi i\hbar t}$.

Now in order to simplify computations let us consider a strict braided monoidal category equivalent to $\mathscr{D}(\mathfrak{g},\hbar)$, which exists by Mac Lane's theorem. We continue to use the same symbols $T_{\mu,\eta}$ and S_μ for morphisms in this new category. We then have

$$(T_{\mu,\eta}\otimes\iota)T_{\mu+\eta,\nu} = (\iota\otimes T_{\eta,\nu})T_{\mu,\eta+\nu} \;\text{ and }\; \sigma T_{\mu,\eta} = q^{(\mu,\eta)}T_{\eta,\mu},$$

while $g^\hbar(\mu,\eta)$ is defined by the identity

$$g^\hbar(\mu,\eta)S_{\mu+\eta} = S_\mu(\iota\otimes S_\eta\otimes\iota)(\bar T_{\mu,\eta}\otimes T_{\mu,\eta}).$$

We have

$$\begin{aligned}
&S_\mu(\iota\otimes S_\eta\otimes\iota)(\iota\otimes\iota\otimes S_\nu\otimes\iota\otimes\iota)(\iota\otimes\bar T_{\eta,\nu}\otimes T_{\nu,\eta}\otimes\iota)(\bar T_{\mu,\eta+\nu}\otimes T_{\eta+\nu,\mu})\\
&\quad= g^\hbar(\eta,\nu)S_\mu(\iota\otimes S_{\eta+\nu}\otimes\iota)(\bar T_{\mu,\eta+\nu}\otimes T_{\eta+\nu,\mu})\\
&\quad= g^\hbar(\eta,\nu)g^\hbar(\mu,\eta+\nu)S_{\mu+\eta+\nu}.
\end{aligned}$$

On the other hand, the same expression equals

$$\begin{aligned}
&S_\mu(\iota\otimes S_\eta\otimes\iota)(\iota\otimes\iota\otimes S_\nu\otimes\iota\otimes\iota)(\bar T_{\mu,\eta}\otimes\iota\otimes\iota\otimes T_{\eta,\mu})(\bar T_{\mu+\eta,\nu}\otimes T_{\nu,\mu+\eta})\\
&\quad= S_\mu(\iota\otimes S_\eta\otimes\iota)(\bar T_{\mu,\eta}\otimes T_{\eta,\mu})(\iota\otimes S_\nu\otimes\iota)(\bar T_{\mu+\eta,\nu}\otimes T_{\nu,\mu+\eta})\\
&\quad= g^\hbar(\mu,\eta)g^\hbar(\mu+\eta,\nu)S_{\mu+\eta+\nu}.
\end{aligned}$$

This proves the cocycle identity

$$g^\hbar(\mu,\eta)g^\hbar(\mu+\eta,\nu) = g^\hbar(\eta,\nu)g^\hbar(\mu,\eta+\nu).$$

It remains to show that $g^\hbar(\mu,\eta) = g^\hbar(\eta,\mu)$. By the hexagon identities $\sigma_{12,3} = (\sigma\otimes\iota)(\iota\otimes\sigma)$ and $\sigma^{-1}_{1,23} = (\sigma^{-1}\otimes\iota)(\iota\otimes\sigma^{-1})$ we have

$$\sigma^{-1}\otimes\sigma = (\sigma^{-1}_{1,23}\otimes\iota)(\iota\otimes\sigma_{12,3}).$$

Using this and the identity $\sigma T_{\mu,\eta} = q^{(\mu,\eta)} T_{\eta,\mu}$ we compute:

$$\begin{aligned} g^{\hbar}(\mu,\eta) S_{\mu+\eta} &= S_\mu(\iota\otimes S_\eta\otimes\iota)(\sigma^{-1}_{1,23}\otimes\iota)(\iota\otimes\sigma_{12,3})(\bar T_{\eta,\mu}\otimes T_{\mu,\eta}) \\ &= S_\mu(S_\eta\otimes\iota\otimes\iota)(\iota\otimes\sigma_{12,3})(\bar T_{\eta,\mu}\otimes T_{\mu,\eta}) \\ &= S_\eta(\iota\otimes\iota\otimes S_\mu)(\iota\otimes\sigma_{12,3})(\bar T_{\eta,\mu}\otimes T_{\mu,\eta}) \\ &= S_\eta(\iota\otimes S_\mu\otimes\iota)(\bar T_{\eta,\mu}\otimes T_{\mu,\eta}) \\ &= g^{\hbar}(\eta,\mu) S_{\mu+\eta}. \end{aligned}$$

Hence $g^{\hbar}(\mu,\eta) = g^{\hbar}(\eta,\mu)$. □

By Lemma 3.4.5 for generic $\hbar$ the cocycle $g^{\hbar}(\mu,\eta)$ is a coboundary, so we can write $g^{\hbar}(\mu,\eta) = g^{\hbar}(\mu+\eta)g^{\hbar}(\mu)^{-1}g^{\hbar}(\eta)^{-1}$. Furthermore, the values $g^{\hbar}(\omega_i) \in \mathbb{C}^*$ can be chosen arbitrary and they completely determine $g^{\hbar}(\mu)$. We make such a choice so that $g^{\hbar}(\omega_i)$ depends analytically on $\hbar$ and $g^0(\omega_i) = 1$. In this case, for every $\mu \in P_+$, the function $\hbar \mapsto g^{\hbar}(\mu)$ extends to an analytic function on $\mathbb{C}$ excluding a closed discrete subset. Put

$$S^{\hbar}_\mu = g^{\hbar}(\mu) S_\mu$$

and define

$$\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\mu}\colon \bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta} \to \bar V_\mu \otimes V_{\lambda+\mu}$$

as the composition

$$\bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta} \xrightarrow{\bar T_{\mu,\eta}\otimes T_{\eta,\lambda+\mu}} (\bar V_\mu\otimes \bar V_\eta)\otimes(V_\eta\otimes V_{\lambda+\mu}) \xrightarrow{(\iota\otimes S^{\hbar}_\eta\otimes\iota)B} \bar V_\mu\otimes V_{\lambda+\mu}$$

where $B = (\iota\otimes\Phi^{-1})\Phi_{1,2,34}$. By the choice of $g^{\hbar}(\mu)$ the morphisms $\mathrm{tr}^{\eta,\hbar}_{\mu,\mu}$ form a projective system. It is easy to check that the same is true for the morphisms $\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\mu}$ for every $\lambda\in P$. Therefore similarly to Section 3.3 we can define pro-objects

$$M^{\hbar}_\lambda = \{\bar V_\mu\otimes V_{\lambda+\mu}\}_\mu \ \text{ and } \ M^{\hbar} = \{\oplus_{\lambda\in X}\bar V_\mu\otimes V_{\lambda+\mu}\}_{X,\mu}$$

in pro-$\mathscr{D}(\mathfrak{g},\hbar)$. They are defined for generic $\hbar$. For $\hbar = 0$ these are the pro-objects M_λ and M from Section 3.3 in the case $q = 1$.

Lemma 4.2.3. — *For generic $\hbar$ and any $\lambda\in P$ the pro-object $M^{\hbar}_\lambda$ considered as a pro-object of $\mathscr{D}(\mathfrak{g},0) = \mathscr{C}_1(\mathfrak{g})$ is isomorphic to M_λ.*

Proof. — Note that the maps $\mathrm{tr}^{\eta,\hbar}_{\mu,\lambda+\mu}$ are surjective for generic $\hbar$, since this is true for $\hbar = 0$, and for fixed μ, η and λ they are analytic in $\hbar$ outside a closed discrete subset of $\mathbb{C}$. For every such $\hbar$ fix a regular weight $\mu\in P_+$ (that is, $\mu(i) > 0$ for all i) such that $\lambda+\mu\in P_+$. Then inductively define isomorphisms

$$f_n\colon \bar V_{n\mu}\otimes V_{\lambda+n\mu} \to \bar V_{n\mu}\otimes V_{\lambda+n\mu}$$

such that $f_1 = \iota$ and the diagrams

$$\begin{tikzcd}
\bar V_{n\mu} \otimes V_{\lambda+n\mu} \arrow[r, "\mathrm{tr}^\hbar"] \arrow[d, "f_n \wr"'] & \bar V_{(n-1)\mu} \otimes V_{\lambda+(n-1)\mu} \arrow[d, "\wr f_{n-1}"] \\
\bar V_{n\mu} \otimes V_{\lambda+n\mu} \arrow[r, "\mathrm{tr}"'] & \bar V_{(n-1)\mu} \otimes V_{\lambda+(n-1)\mu}
\end{tikzcd}$$

commute. Such morphisms are easy to find using right inverses to tr and $\mathrm{tr}^\hbar$. They define the required isomorphism $M^\hbar_\lambda \to M_\lambda$. □

Since M represents the forgetful functor, we therefore see that for generic $\hbar$ the functor $F^\hbar = \mathrm{Hom}(M^\hbar, \cdot)$ on $\mathscr{D}(\mathfrak{g}, \hbar)$ is isomorphic to the forgetful functor, so that $\dim F^\hbar(V) = \dim V$. The main point of using $M^\hbar$ is that it has a structure of a comonoid and therefore $F^\hbar$ becomes a tensor functor. Namely, define

$$\delta^\hbar_{\lambda_1,\lambda_2} \colon M^\hbar_{\lambda_1+\lambda_2} \to M^\hbar_{\lambda_1} \otimes M^\hbar_{\lambda_2}$$

using morphisms $m^\hbar_{\mu_1,\mu_2,\lambda_1,\lambda_2}$ given as the compositions

$$\bar V_{\mu_1+\mu_2} \otimes V_{\lambda_1+\lambda_2+\mu_1+\mu_2} \xrightarrow{\bar T_{\mu_1,\mu_2} \otimes T_{\lambda_1+\mu_1,\lambda_2+\mu_2}} (\bar V_{\mu_1} \otimes \bar V_{\mu_2}) \otimes (V_{\lambda_1+\mu_1} \otimes V_{\lambda_2+\mu_2})$$
$$\xrightarrow{q^{(\lambda_1+\mu_1,\mu_2)} B^{-1}(\iota\otimes\sigma\otimes\iota)B} (\bar V_{\mu_1} \otimes V_{\lambda_1+\mu_1}) \otimes (\bar V_{\mu_2} \otimes V_{\lambda_2+\mu_2}),$$

where $B = (\iota \otimes \Phi^{-1})\Phi_{1,2,34}$. We remark that it does require some effort to prove that $m^\hbar$ are compatible with $\mathrm{tr}^\hbar$ and so define a morphism of pro-objects. They also define a morphism $\delta^\hbar \colon M^\hbar \to M^\hbar \otimes M^\hbar$ that can be thought of as $\sum_{\lambda_1,\lambda_2} \delta_{\lambda_1,\lambda_2}$.

Let also $\varepsilon^\hbar \colon M^\hbar \to V_0 = \mathbb{C}$ be the morphism defined by the morphisms

$$S^\hbar_\mu \colon \bar V_\mu \otimes V_\mu \to \mathbb{C}.$$

Lemma 4.2.4. — *The triple $(M^\hbar, \delta^\hbar, \varepsilon^\hbar)$ is a comonoid in* pro-$\mathscr{D}(\mathfrak{g}, \hbar)$.

Proof. — Let us only check the coassociativity of $\delta^\hbar$. For this it suffices to show that

$$\Phi(\delta^\hbar_{\lambda_1,\lambda_2} \otimes \iota)\delta^\hbar_{\lambda_1+\lambda_2,\lambda_3} = (\iota \otimes \delta^\hbar_{\lambda_2,\lambda_3})\delta^\hbar_{\lambda_1,\lambda_2+\lambda_3}.$$

As in the proof of Lemma 4.2.2 let us strictify the category $\mathscr{D}(\mathfrak{g}, \hbar)$. In the new strict category it suffices to check that

$$(\iota \otimes \sigma \otimes \iota \otimes \iota)(\bar T \otimes T \otimes \iota \otimes \iota)(\iota \otimes \sigma \otimes \iota)(\bar T \otimes T)$$
$$= (\iota \otimes \iota \otimes \iota \otimes \sigma \otimes \iota)(\iota \otimes \iota \otimes \bar T \otimes T)(\iota \otimes \sigma \otimes \iota)(\bar T \otimes T)$$

as morphisms

$$\bar V_{\mu_1+\mu_2+\mu_3} \otimes V_{\lambda_1+\lambda_2+\lambda_3+\mu_1+\mu_2+\mu_3} \to \bar V_{\mu_1} \otimes V_{\lambda_1+\mu_1} \otimes \bar V_{\mu_2} \otimes V_{\lambda_2+\mu_2} \otimes \bar V_{\mu_3} \otimes V_{\lambda_3+\mu_3},$$

where we omitted the subindices of T and $\bar{T}$, as there is only one choice making the above identity meaningful. The left hand side of that identity is equal to

$$(\iota\otimes\sigma\otimes\iota\otimes\iota\otimes\iota)(\iota\otimes\iota\otimes\sigma_{1,23}\otimes\iota)(\bar{T}\otimes\iota\otimes T\otimes\iota)(\bar{T}\otimes T),$$

whereas the right hand side is equal to

$$(\iota\otimes\iota\otimes\iota\otimes\sigma\otimes\iota)(\iota\otimes\sigma_{12,3}\otimes\iota\otimes\iota)(\iota\otimes\bar{T}\otimes\iota\otimes T)(\bar{T}\otimes T).$$

As $(\bar{T}\otimes\iota)\bar{T}=(\iota\otimes\bar{T})\bar{T}$ and $(T\otimes\iota)T=(\iota\otimes T)T$, we therefore only have to check that

$$(\sigma\otimes\iota\otimes\iota)(\iota\otimes\sigma_{1,23})=(\sigma_{12,3}\otimes\iota)(\iota\otimes\iota\otimes\sigma).$$

But this is immediate from the hexagon identities $\sigma_{1,23}=(\iota\otimes\sigma)(\sigma\otimes\iota)$ and $\sigma_{12,3}=(\sigma\otimes\iota)(\iota\otimes\sigma)$. □

We can therefore define natural maps

$$F_2^\hbar(U,V)\colon F^\hbar(U)\otimes F^\hbar(V)\to F^\hbar(U\otimes V),\quad f\otimes g\mapsto(f\otimes g)\delta^\hbar,$$

which would make $F^\hbar$ a tensor functor provided they were isomorphisms.

Lemma 4.2.5. — *For generic $\hbar$ the maps $F_2^\hbar(U,V)$ are isomorphisms for any finite dimensional $\mathfrak{g}$-modules U and V.*

Proof. — It suffices to check that for all sufficiently large μ_1,μ_2 the map

$$\bigoplus_{\lambda_1,\lambda_2}\mathrm{Hom}(\bar{V}_{\mu_1}\otimes V_{\lambda_1+\mu_1},U)\otimes\mathrm{Hom}(\bar{V}_{\mu_2}\otimes V_{\lambda_2+\mu_2},V)$$
$$\to\mathrm{Hom}(\bar{V}_{\mu_1+\mu_2}\otimes V_{\lambda+\mu_1+\mu_2},U\otimes V),\quad f\otimes g\mapsto(f\otimes g)m^\hbar,$$

where the summation is over weights λ_1 of U and weights λ_2 of V, is an isomorphism for generic $\hbar$. This is indeed the case, since this map is an isomorphism for $\hbar=0$ by results of Section 3.3 (for $q=1$), and it is analytic in $\hbar$ outside a closed discrete set. □

Let us summarize what we have proved so far.

Proposition 4.2.6. — *For generic $\hbar$ the functor $F^\hbar=\mathrm{Hom}(M^\hbar,\cdot)$ is a fiber functor on $\mathscr{D}(\mathfrak{g},\hbar)$ such that* $\dim F^h(V)=\dim V$ *for any finite dimensional $\mathfrak{g}$-module V.*

Observe that this did not require much information about $\Phi(\hbar t_{12},\hbar t_{23})$. Namely, the only properties that we used were that $\Phi(\hbar t_{12},\hbar t_{23})\,|_{V_\mu\otimes V_\eta\otimes V_\nu}$ is analytic outside a closed discrete set, with value 1 at $\hbar=0$, and that $\Phi(\hbar t_{12},\hbar t_{23})$ acts trivially on $\xi_\mu\otimes\xi_\eta\otimes\xi_\nu$.

We next want to define an action of $U_q\mathfrak{g}$ on $F^\hbar$. We proceed as in Section 3.3, but will be more sketchy, referring the reader to the paper by Kazhdan and Lusztig [**49**] or to [**68**] for details.

We want to define an action of $(U_q\mathfrak{g})^{\mathrm{op}}$ on $M^\hbar$. Recalling how it was done in the proof of Theorem 3.3.7, we use the morphisms

$$\tau_{i;\mu,\eta}\colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta,\ \ \xi_{\mu+\eta-\alpha_i} \mapsto \mu(i)\xi_\mu \otimes F_i\xi_\eta - \eta(i)F_i\xi_\mu \otimes \xi_\eta$$

and then attempt to define morphisms $\tilde{F}_i\colon M^\hbar_\lambda \to M^\hbar_{\lambda+\alpha_i}$ using suitably normalized compositions

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{T}_{\mu,\eta}\otimes\tau_{i;\eta,\lambda+\alpha_i+\mu}} (\bar{V}_\mu \otimes \bar{V}_\eta) \otimes (V_\eta \otimes V_{\lambda+\alpha_i+\mu}) \tag{4.2.1}$$
$$\xrightarrow{(\iota\otimes S^\hbar_\eta\otimes\iota)B} \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu},$$

where $B = (\iota \otimes \Phi^{-1})\Phi_{1,2,34}$. Trying to check when such normalizations become compatible with $\mathrm{tr}^\hbar$ one quickly realizes that one needs a relation between the morphisms $\Phi(T\otimes\iota)\tau_i$ and $(\iota\otimes T)\tau_i$. Equations (3.4.1)-(3.4.2) show what we should expect. Namely, it is not difficult to see that if the theorem holds, then there must exist normalizations $\tau^\hbar_i$ of τ_i such that

$$\begin{aligned}[\eta(i)]_{q_i}\Phi(T_{\mu,\eta} \otimes \iota)\tau^\hbar_{i;\mu+\eta,\nu} - [\nu(i)]_{q_i}\Phi(\tau^\hbar_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}& \\ = [\mu(i) + \eta(i)]_{q_i}(\iota \otimes \tau^\hbar_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}&,\end{aligned} \tag{4.2.2}$$

$$\begin{aligned}[\eta(i)]_{q_i}(\iota \otimes T_{\eta,\nu})\tau^\hbar_{i;\mu,\eta+\nu} - [\mu(i)]_{q_i}(\iota \otimes \tau^\hbar_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}& \\ = [\eta(i) + \nu(i)]_{q_i}\Phi(\tau^\hbar_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}&.\end{aligned} \tag{4.2.3}$$

In order to find these normalizations we have to compute the action of Φ on the two-dimensional space of highest weight vectors in the isotypic component of $V_\mu \otimes V_\eta \otimes V_\nu$ corresponding to the weight $\mu + \eta + \nu - \alpha_i$. Thus we have to compute $\Phi(A, B)$ for two-by-two matrices A and B. In this case solutions of the equation

$$w' = \left(\frac{A}{x} + \frac{B}{x-1}\right) w$$

can be written in terms of the Euler-Gauss hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; x)$, which is the unique solution of

$$x(1-x)u'' + (\gamma - (\alpha+\beta+1)x)u' - \alpha\beta u = 0$$

that is analytic in a neighborhood of 0 and equals 1 at $x = 0$. Using known relations for the hypergeometric function it can then be shown that identities (4.2.2)-(4.2.3) are indeed satisfied with

$$\tau^\hbar_{i;\mu,\eta} = \frac{\tau_{i;\mu,\eta}}{\Gamma(1+\hbar d_i\mu(i))\Gamma(1+\hbar d_i\eta(i))\Gamma(1-\hbar d_i(\mu(i)+\eta(i)))}.$$

This is essentially the only part of the proof, for which we need to know how $\Phi(\hbar t_{12}, \hbar t_{23})$ is defined.

Using (4.2.2)-(4.2.3) and working in a strictified category of $\mathscr{D}(\mathfrak{g},\hbar)$ one can then check that compositions (4.2.1), with τ_i replaced by $\tau_i^\hbar$, define morphisms $\tilde F_i\colon M^\hbar_\lambda\to M^\hbar_{\lambda+\alpha_i}$. These morphisms define a morphism $M^\hbar\to M^\hbar$, which we continue to denote by $\tilde F_i$. Similarly we define $\tilde E_i\colon M^\hbar\to M^\hbar$, see the proof of Theorem 3.3.7, as well as morphisms $\tilde K_i$ that act on $M^\hbar_\lambda$ as the scalars $q_i^{\lambda(i)}$. It is then not difficult to prove the following.

Proposition 4.2.7. — *The endomorphisms $\tilde E_i$, $\tilde F_i$ and $\tilde K_i$ of $M^\hbar$ satisfy the relations*

$$\tilde E_j\tilde K_i=q_i^{a_{ij}}\tilde K_i\tilde E_j,\quad \tilde F_j\tilde K_i=q_i^{-a_{ij}}\tilde K_i\tilde F_j,$$

$$\tilde F_j\tilde E_i-\tilde E_i\tilde F_j=\delta_{ij}f_i(\hbar)\frac{\tilde K_i-\tilde K_i^{-1}}{q_i-q_i^{-1}},$$

where $f_i(\hbar)$ is defined by the identity

$$S^\hbar_{\omega_i}(\iota\otimes S^\hbar_{\omega_i}\otimes\iota)(\iota\otimes\Phi^{-1})\Phi_{1,2,34}(\bar\tau^\hbar_{i;\omega_i,\omega_i}\otimes\tau^\hbar_{i;\omega_i,\omega_i})=-[2]_{q_i}f_i(\hbar)S^\hbar_{2\omega_i-\alpha_i}.$$

In particular, f_i is analytic outside a closed discrete set, and $f_i(0)=1$.

In order to define an action of $(U_q\mathfrak{g})^{\mathrm{op}}$ on $M^\hbar$ one has to prove more complicated relations for $\tilde E_i$ and $\tilde F_i$. As we will see shortly, fortunately, this is not needed to finish the proof of the theorem.

Proof of Theorem 4.2.1 for generic $\hbar$. — We already know that for generic $\hbar$ the functor $F^\hbar=\mathrm{Hom}(M^\hbar,\cdot)$ is a fiber functor such that $\dim F^\hbar(V)=\dim V$. If in addition $f_i(\hbar)\neq 0$, where f_i is from the previous proposition, we define an action of $U_q\mathfrak{g}$ on $F^\hbar(V)$ by

$$E_i^qf=f_i(\hbar)^{-1}f\tilde E_i,\quad F_i^qf=f\tilde F_i,\quad K_i^qf=f\tilde K_i.$$

In order to check that this is indeed an action for generic $\hbar$, we have to show that the element

$$G^q_{ij}=\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}(E^q_i)^kE^q_j(E^q_i)^{1-a_{ij}-k},$$

as well as the similar element for F^q_i, is zero. It suffices to check how G^q_{ij} acts on $F^\hbar(V_\lambda)$ for every $\lambda\in P_+$.

The morphisms $\mathrm{tr}^{\mu,\hbar}_{0,\lambda}\colon\bar V_\mu\otimes V_{\lambda+\mu}\to V_0\otimes V_\lambda=V_\lambda$ define a morphism $\xi^\hbar_\lambda\colon M^\hbar\to V_\lambda$, that is, a vector in $F^\hbar(V_\lambda)$. We have $E^q_i\xi^\hbar_\lambda=0$, as there are no nonzero morphisms $\bar V_\mu\otimes V_{\lambda+\alpha_i+\mu}\to V_\lambda$. In particular, $G^q_{ij}\xi^\hbar_\lambda=0$. Using the relations $E^q_kF^q_l-F^q_lE^q_k=\delta_{kl}(K^q_k-(K^q_k)^{-1})/(q_k-q_k^{-1})$, $K^q_kE^q_l=q_k^{a_{kl}}E^q_lK^q_k$ and $K^q_kF^q_l=q_k^{-a_{kl}}F^q_lK^q_k$ it can be easily checked that G^q_{ij} commutes with F^q_l for all l. Therefore to prove that $G^q_{ij}=0$ on $F^\hbar(V_\lambda)$ it suffices to show that $F^\hbar(V_\lambda)$ is spanned by vectors $F^q_{i_1}\dots F^q_{i_m}\xi^\hbar_\lambda$. For this choose a finite set I of multi-indices $(i_1,\dots,i_m)$ such that the vectors $F_{i_1}\dots F_{i_m}\xi_\lambda$ form a basis in V_λ.

Since $\dim F^\hbar(V_\lambda) = \dim V_\lambda$, it then suffices to check that for generic $\hbar$ the vectors $F^q_{i_1}\dots F^q_{i_m}\xi^\hbar_\lambda$ are linearly independent. But these vectors are defined by morphisms

$$\bar{V}_\mu \otimes V_{\lambda-\alpha_{i_1}-\dots-\alpha_{i_m}+\mu} \to V_\lambda.$$

Therefore it suffices to check that these morphisms are linearly independent for generic $\hbar$. Since this is true for $\hbar = 0$, this is indeed the case by analyticity.

Thus for generic $\hbar$ the functor $F^\hbar$ can be considered as a functor from $\mathscr{D}(\mathfrak{g},\hbar)$ into $\mathscr{C}_q(\mathfrak{g})$. Furthermore, from the above argument we see that for generic $\hbar$ the module $F^\hbar(V_\lambda)$ has a nonzero vector of weight λ killed by E^q_i, so V^q_λ embeds into $F^\hbar(V_\lambda)$. Since $\dim V^q_\lambda = \dim V_\lambda = \dim F^\hbar(V_\lambda)$, we conclude that $F^\hbar(V_\lambda)$ is an irreducible $U_q\mathfrak{g}$-module with highest weight λ. It follows that $F^\hbar$ is an equivalence of categories.

It remains to check that $F^\hbar$ is a braided monoidal equivalence, that is, it respects the braiding and $F^\hbar_2(U,V)$ is a morphism of $U_q\mathfrak{g}$-modules. For the first property, in view of Definition (2.6.1) of the R-matrix for $U_q\mathfrak{g}$, it suffices to show that

$$\sigma(\bar{\xi}^\hbar_\lambda \otimes \xi^\hbar_\mu)\delta^\hbar = q^{-(\lambda,\mu)}(\xi^\hbar_\mu \otimes \bar{\xi}^\hbar_\lambda)\delta^\hbar$$

as morphisms $M^\hbar \to V_\mu \otimes \bar{V}_\lambda$, where $\bar{\xi}^\hbar_\lambda \in F^\hbar(\bar{V}_\lambda)$ is the lowest weight vector defined using the morphisms $\operatorname{tr}^{\eta,\hbar}_{\lambda,0}\colon \bar{V}_{\lambda+\eta} \otimes V_\eta \to \bar{V}_\lambda \otimes V_0 = \bar{V}_\lambda$. This is more or less immediate by definition of $\delta^\hbar$. For the second property it suffices to check that

$$\delta^\hbar\tilde{E}_i = (\tilde{E}_i \otimes \iota + \tilde{K}_i \otimes \tilde{E}_i)\delta^\hbar \quad\text{and}\quad \delta^\hbar\tilde{F}_i = (\tilde{F}_i \otimes \tilde{K}_i^{-1} + \iota \otimes \tilde{F}_i)\delta^\hbar.$$

We omit this computation, which is again based on (4.2.2)-(4.2.3). □

References. — [**27**], [**28**], [**33**], [**49**], [**68**].

4.3. DRINFELD TWISTS

Existence of an equivalence of categories $\mathscr{D}(\mathfrak{g},\hbar)$ and $\mathscr{C}_q(\mathfrak{g})$ ($\hbar \in i\mathbb{R}$ and $q = e^{\pi i\hbar}$) can be reformulated in more concrete terms as follows. Recall that by Lemma 3.2.4 there exists an isomorphism $\varphi\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ extending the canonical identification of the centers.

Theorem 4.3.1. — *The categories $\mathscr{D}(\mathfrak{g},\hbar)$ and $\mathscr{C}_q(\mathfrak{g})$ are braided monoidally equivalent, with the equivalence mapping an irreducible $\mathfrak{g}$-module with highest weight λ onto an irreducible $U_q\mathfrak{g}$-module with highest weight λ, if and only if there exists an invertible element $\mathscr{F} \in \mathscr{U}(G\times G)$ such that*

(i) $(\varphi\otimes\varphi)\hat{\Delta}_q = \mathscr{F}\hat{\Delta}\varphi(\cdot)\mathscr{F}^{-1}$;

(ii) $(\hat{\varepsilon}\otimes\iota)(\mathscr{F}) = (\iota\otimes\hat{\varepsilon})(\mathscr{F}) = 1$;

(iii) $(\varphi\otimes\varphi)(\mathscr{R}_q) = \mathscr{F}_{21}q^t\mathscr{F}^{-1}$;

(iv) $(\iota\otimes\hat\Delta)(\mathscr{F}^{-1})(1\otimes\mathscr{F}^{-1})(\mathscr{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{F})=\Phi(\hbar t_{12},\hbar t_{23})$.

Furthermore, assuming that φ is a $*$-isomorphism, the equivalence can be chosen to be unitary if and only if the element $\mathscr{F}$ can be chosen to be unitary.

Proof. — Assume we have a braided equivalence F as in the formulation. We may assume that $F(V_\lambda)=V_\lambda^q$. Choose linear isomorphisms $\eta_{V_\lambda}\colon V_\lambda\to V_\lambda^q$ that implement the isomorphism $\varphi\colon\mathscr{U}(G_q)\to\mathscr{U}(G)$; for $\lambda=0$ we take $\eta_{V_0}=F_0$. The isomorphisms η_{V_λ} define natural isomorphisms $\eta_U\colon U\to F(U)$ for finite dimensional $\mathfrak{g}$-modules U, and we have $\omega\eta_U=\eta_U\varphi(\omega)$ for all $\omega\in\mathscr{U}(G_q)$.

Let $\mathscr{F}\in\mathscr{U}(G\times G)$ be the element defined by requiring commutativity of the diagrams

$$\begin{tikzcd} U\otimes V \arrow[r, "\mathscr{F}^{-1}"] \arrow[d, "\eta\otimes\eta"'] & U\otimes V \arrow[d, "\eta"] \\ F(U)\otimes F(V) \arrow[r, "F_2"'] & F(U\otimes V). \end{tikzcd}$$

In other words, $\mathscr{F}$ acts on $U\otimes V$ as $(\eta_U^{-1}\otimes\eta_V^{-1})F_2(U,V)^{-1}\eta_{U\otimes V}$. For every $\omega\in\mathscr{U}(G_q)$ we then have the following identities for the maps $U\otimes V\to F(U\otimes V)$:

$$F_2(\eta\otimes\eta)(\varphi\otimes\varphi)\hat\Delta_q(\omega)=F_2\hat\Delta_q(\omega)(\eta\otimes\eta)=\omega F_2(\eta\otimes\eta)$$

and

$$(\eta\mathscr{F}^{-1})(\mathscr{F}\hat\Delta\varphi(\omega)\mathscr{F}^{-1})=\omega\eta\mathscr{F}^{-1}.$$

Since $F_2(\eta\otimes\eta)=\eta\mathscr{F}^{-1}$, it follows that $(\varphi\otimes\varphi)\hat\Delta_q(\omega)=\mathscr{F}\hat\Delta\varphi(\omega)\mathscr{F}^{-1}$.

Property (ii) follows easily by definition, using that $\eta_{V_0}=F_0$.

Turning to property (iii), we have the commutative diagram

$$\begin{tikzcd} U\otimes V \arrow[r, "\eta"] \arrow[d, "\Sigma q^t"'] & F(U\otimes V) \arrow[r, "F_2^{-1}"] \arrow[d, "F(\Sigma q^t)"'] & F(U)\otimes F(V) \arrow[r, "\eta^{-1}\otimes\eta^{-1}"] \arrow[d, "\Sigma\mathscr{R}_q"] & U\otimes V \arrow[d, "\Sigma(\varphi\otimes\varphi)(\mathscr{R}_q)"] \\ V\otimes U \arrow[r, "\eta"] & F(V\otimes U) \arrow[r, "F_2^{-1}"] & F(V)\otimes F(U) \arrow[r, "\eta^{-1}\otimes\eta^{-1}"] & V\otimes U. \end{tikzcd}$$

Therefore $\Sigma(\varphi\otimes\varphi)(\mathscr{R}_q)=\mathscr{F}\Sigma q^t\mathscr{F}^{-1}$, that is, $(\varphi\otimes\varphi)(\mathscr{R}_q)=\mathscr{F}_{21}q^t\mathscr{F}^{-1}$.

Finally, we also have the commutative diagrams

$$\begin{tikzcd} (F(U)\otimes F(V))\otimes F(W) \arrow[r, "F_2\otimes\iota"] \arrow[d, "\iota"'] & F(U\otimes V)\otimes F(W) \arrow[r, "F_2"] & F((U\otimes V)\otimes W) \arrow[d, "F(\Phi)"] \\ F(U)\otimes(F(V)\otimes F(W)) \arrow[r, "\iota\otimes F_2"] & F(U)\otimes F(V\otimes W) \arrow[r, "F_2"] & F(U\otimes(V\otimes W)), \end{tikzcd}$$

where $\Phi = \Phi(\hbar t_{12}, \hbar t_{23})$. Using that $F_2 = \eta\mathscr{F}^{-1}(\eta^{-1}\otimes\eta^{-1})$ we get

$$F(\Phi) = \eta(\iota\otimes\hat\Delta)(\mathscr{F}^{-1})(1\otimes\mathscr{F}^{-1})(\mathscr{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{F})\eta^{-1}.$$

As $\eta^{-1}F(\Phi)\eta = \Phi$ by naturality of η, this is exactly property (iv).

If in addition F was unitary and φ was a $*$-isomorphism, we could choose η_{V_λ} to be unitary, and then $\mathscr{F}$ would be unitary as well.

Conversely, assume we are given $\mathscr{F}$ with properties (i)-(iv). Using φ we can consider any $\mathfrak{g}$-module as a $U_q\mathfrak{g}$-module, so we have a functor $F\colon \mathscr{D}(\mathfrak{g},\hbar)\to\mathscr{C}_q(\mathfrak{g})$. We make it a tensor functor by letting $F_0=\iota$ and $F_2=\mathscr{F}^{-1}$. It is easy to check that F becomes a braided monoidal equivalence. $\square$

Note that by Lemma 3.2.4 we already know that there exists $\mathscr{F}\in\mathscr{U}(G\times G)$ with property (i). Property (ii) is a simple normalization condition. The existence of $\mathscr{F}$ with property (iii) is also not difficult to show using Equation (2.6.6) for the R-matrix and the identity

$$t = \frac{1}{2}(\hat\Delta(C) - 1\otimes C - C\otimes 1),$$

where $C\in U\mathfrak{g}$ is the Casimir operator. Therefore the crucial property is number (iv).

Definition 4.3.2. — An invertible element $\mathscr{F}\in\mathscr{U}(G\times G)$ with properties (i)-(iv) from Theorem 4.3.1 is called a Drinfeld twist for G_q corresponding to the isomorphism $\varphi\colon\mathscr{U}(G_q)\to\mathscr{U}(G)$.

Therefore the main result of the previous section can be formulated by saying that for generic $q>0$ there exists a Drinfeld twist for G_q.

Lemma 4.3.3. — *If the isomorphism $\varphi\colon\mathscr{U}(G_q)\to\mathscr{U}(G)$ is $*$-preserving and there exists a Drinfeld twist $\mathscr{E}$ corresponding to φ, then the unitary $\mathscr{F}$ in the polar decomposition $\mathscr{E} = \mathscr{F}|\mathscr{E}|$ is a unitary Drinfeld twist corresponding to φ.*

Proof. — Indeed, since $\hat\Delta_q$, $\hat\Delta$ and φ are $*$-homomorphisms, condition (i) on $\mathscr{E}$ implies that

$$\mathscr{E}\hat\Delta\varphi(\cdot)\mathscr{E}^{-1} = (\mathscr{E}^{-1})^*\hat\Delta\varphi(\cdot)\mathscr{E}^*,$$

that is, the element $\mathscr{E}^*\mathscr{E}\in\mathscr{U}(G\times G)$ is invariant. It follows that $|\mathscr{E}|$ is also invariant. Hence

$$\mathscr{E}\hat\Delta\varphi(\cdot)\mathscr{E}^{-1} = \mathscr{F}|\mathscr{E}|\hat\Delta\varphi(\cdot)|\mathscr{E}|^{-1}\mathscr{F}^{-1} = \mathscr{F}\hat\Delta\varphi(\cdot)\mathscr{F}^{-1},$$

so condition (i) for $\mathscr{F}$ is satisfied. Condition (ii) is also obviously satisfied. Turning to (iii), recall that the R-matrix has the property $\mathscr{R}_q^* = (\mathscr{R}_q)_{21}$. So applying the $*$-operation and then the flip to the identity $(\varphi\otimes\varphi)(\mathscr{R}_q) = \mathscr{E}_{21}q^t\mathscr{E}^{-1}$ we get

$$(\varphi\otimes\varphi)(\mathscr{R}_q) = (\mathscr{E}^{-1})^*_{21}q^t\mathscr{E}^*.$$

Therefore $(\mathscr{E}^*\mathscr{E})_{21}q^t = q^t\mathscr{E}^*\mathscr{E}$ and hence $|\mathscr{E}|_{21}q^t = q^t|\mathscr{E}|$. It follows that

$$(\varphi\otimes\varphi)(\mathscr{R}_q) = \mathscr{E}_{21}q^t\mathscr{E}^{-1} = \mathscr{F}_{21}|\mathscr{E}|_{21}q^t|\mathscr{E}|^{-1}\mathscr{F}^{-1} = \mathscr{F}_{21}q^t\mathscr{F}^{-1}.$$

It remains to check (iv). As usual write Φ for $\Phi(\hbar t_{12},\hbar t_{23})$. Consider the element

$$\Phi_0 = (\iota\otimes\hat\Delta)(\mathscr{F}^{-1})(1\otimes\mathscr{F}^{-1})(\mathscr{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{F}).$$

We have to show that $\Phi_0 = \Phi$. Using that $|\mathscr{E}|$ is invariant one easily checks that

$$\Phi = (\iota\otimes\hat\Delta)(|\mathscr{E}|^{-1})(1\otimes|\mathscr{E}|^{-1})\Phi_0(|\mathscr{E}|\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|). \tag{4.3.1}$$

Since Φ_0 is defined by the unitary element $\mathscr{F}$, it is itself unitary. Since Φ is also unitary, taking the inverses in the above identity and then applying the $*$-operation we get

$$\Phi = (\iota\otimes\hat\Delta)(|\mathscr{E}|)(1\otimes|\mathscr{E}|)\Phi_0(|\mathscr{E}|^{-1}\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|^{-1}).$$

Therefore

$$\begin{aligned}(\iota\otimes\hat\Delta)(|\mathscr{E}|^{-1})(1\otimes|\mathscr{E}|^{-1})\Phi_0(|\mathscr{E}|\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|)&\\ = (\iota\otimes\hat\Delta)(|\mathscr{E}|)(1\otimes|\mathscr{E}|)\Phi_0(|\mathscr{E}|^{-1}\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|^{-1}).&\end{aligned}$$

Since $|\mathscr{E}|$ is invariant, the positive operators $(\iota\otimes\hat\Delta)(|\mathscr{E}|)$ and $1\otimes|\mathscr{E}|$, as well as $|\mathscr{E}|\otimes 1$ and $(\hat\Delta\otimes\iota)(|\mathscr{E}|)$, commute. So we can write

$$\Phi_0((|\mathscr{E}|\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|))^2 = ((1\otimes|\mathscr{E}|)(\iota\otimes\hat\Delta)(|\mathscr{E}|))^2\Phi_0.$$

Consequently

$$\Phi_0(|\mathscr{E}|\otimes 1)(\hat\Delta\otimes\iota)(|\mathscr{E}|) = (1\otimes|\mathscr{E}|)(\iota\otimes\hat\Delta)(|\mathscr{E}|)\Phi_0,$$

and returning to (4.3.1) we get $\Phi = \Phi_0$. □

Thus for generic $q>0$ there exists a unitary Drinfeld twist. In order to extend this to all $q>0$ we will study continuity properties of Drinfeld twists as functions of the deformation parameter.

For every $q>0$ choose a $*$-isomorphism $\varphi^q\colon \mathscr{U}(G_q)\to\mathscr{U}(G)$ extending the canonical identification of the centers. We say that the family $\{\varphi^q\}_{q>0}$ is continuous, if the elements $\varphi^q(E_i^q)$, $\varphi^q(F_i^q)$ and $\varphi^q(H_i)$, $1\le i\le r$, depend continuously on q in the weak* topology on $\mathscr{U}(G) = \mathbb{C}[G]^*$; recall that we consider the maximal torus T as subset of $\mathscr{U}(G_q)$ such that $K_i^q = q_i^{H_i}$ for $q\ne 1$.

Lemma 4.3.4. — *A continuous family of $*$-isomorphisms $\varphi^q\colon \mathscr{U}(G_q)\to\mathscr{U}(G)$ always exists. Furthermore, we may assume that φ^q is the identity map on T.*

Proof. — It suffices to show that for every $\lambda\in P_+$ there exist unitaries $u_\lambda^q\colon V_\lambda^q\to V_\lambda$ that map $V_\lambda^q(\mu)$ onto $V_\lambda(\mu)$ and are such that the operators $u_\lambda^q\pi_\lambda^q(E_i^q)u_\lambda^{q*}$ and $u_\lambda^q\pi_\lambda^q(F_i^q)u_\lambda^{q*}$ depend continuously on q. It is enough to show that such unitaries exist locally.

Therefore fix λ and $q_0 > 0$. For every multi-index $I = (i_1, \dots, i_k)$ $(1 \le i_j \le r)$ put $e^q_I = F^q_{i_1} \dots F^q_{i_k} \xi^q_\lambda \in V^q_\lambda$. We can choose multi-indices $I_1, \dots, I_n$ such that the vectors $e^{q_0}_{I_1}, \dots, e^{q_0}_{I_n}$ form a basis in $V^{q_0}_\lambda$. The relations in $U_q\mathfrak{g}$ together with the identities $F^{q*}_i = (K^q_i)^{-1} E^q_i$ imply that the scalar products (e^q_I, e^q_J) depend continuously on q. Hence, for some $\varepsilon > 0$, the vectors $e^q_{I_1}, \dots, e^q_{I_n}$ form a basis in V^q_λ for all $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$. Applying the Gram-Schmidt orthogonalization we get an orthonormal basis $\zeta^q_1, \dots, \zeta^q_n$ in V^q_λ. Since the weight spaces are mutually orthogonal and the vectors e^q_I are weight vectors, this basis consists of weight vectors. Let $v^q \colon V^q_\lambda \to V^{q_0}_\lambda$ be the unitary mapping ζ^q_i into $\zeta^{q_0}_i$. By construction it maps $V^q_\lambda(\mu)$ onto $V^{q_0}_\lambda(\mu)$. Furthermore, for every multi-index I, the coefficients of e^q_I in the basis $\zeta^q_1, \dots, \zeta^q_n$ depend continuously on q, hence the matrix coefficients of $\pi^q_\lambda(F^q_i)$ in this basis also depend continuously on q. It follows that the operators $v^q \pi^q_\lambda(F^q_i) v^{q*}$ depend continuously on q. Since $v^q \pi^q_\lambda(H_i) v^{q*} = \pi^{q_0}_\lambda(H_i)$ and $E^q_i = q^{d_i H_i} F^{q*}_i$, the operators $v^q \pi^q_\lambda(E^q_i) v^{q*}$ also depend continuously on q.

Now take an arbitrary unitary $u \colon V^{q_0}_\lambda \to V_\lambda$ that maps $V^{q_0}_\lambda(\mu)$ onto $V_\lambda(\mu)$. Then the unitaries $u^q_\lambda = uv^q$, $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$, have the required properties. □

We can now easily finish the proof of Theorem 4.2.1.

Proposition 4.3.5. — *Assume $\{\varphi^q \colon \mathscr{U}(G_q) \to \mathscr{U}(G)\}_q$ is a continuous family of $*$-isomorphisms. Then the set of pairs $(q, \mathscr{F}^q)$, where $\mathscr{F}^q$ is a unitary Drinfeld twist for G_q corresponding to φ^q, is closed in $\mathbb{R}^*_+ \times \mathscr{U}(G \times G)$. In particular, for every $q > 0$ there exists a unitary Drinfeld twist for G_q corresponding to φ^q.*

Proof. — By definition of the R-matrix $\mathscr{R}_q$ and the coproduct $\hat\Delta_q$, it is easy to see that the elements $(\varphi^q \otimes \varphi^q)(\mathscr{R}_q)$ and $(\varphi^q \otimes \varphi^q)\hat\Delta_q(X^q)$, where $X^q = E^q_i, F^q_i, H_i$, depend continuously on q. This immediately gives the first statement in the formulation. Since the set of unitary elements in $\mathscr{U}(G \times G)$ is compact and we already know that for generic $q > 0$ there exists a unitary Drinfeld twist for G_q, the second statement also follows. □

As follows from Theorem 4.3.1, existence of a Drinfeld twist for G_q is independent of the choice of an isomorphism $\mathscr{U}(G_q) \cong \mathscr{U}(G)$. Explicitly this can be seen as follows. Assume φ and ψ are two isomorphisms $\mathscr{U}(G_q) \to \mathscr{U}(G)$ extending the canonical identification of the centers, and $\mathscr{F}$ is a Drinfeld twist corresponding to φ. There exists an invertible element $u \in \mathscr{U}(G)$ such that $\psi = (\operatorname{Ad} u)\varphi$ and $\hat\varepsilon(u) = 1$; furthermore, if φ and ψ are $*$-preserving, then u can be chosen to be unitary. Then one can easily check that the element

$$(u \otimes u)\mathscr{F}\hat\Delta(u)^{-1}$$

is a Drinfeld twist corresponding to ψ.

In the case $\psi = \varphi$ this clearly shows that a Drinfeld twist corresponding to a fixed isomorphism φ is not unique: for any invertible central element $c \in \mathscr{U}(G)$ with $\hat\varepsilon(c) = 1$

we get a new Drinfeld twist $(c\otimes c)\mathscr{F}\hat\Delta(c)^{-1}$. It turns out, this is the only way to construct new Drinfeld twists.

Theorem 4.3.6. — *Suppose $\mathscr{F}$ and $\mathscr{F}'$ are two Drinfeld twists for G_q corresponding to the same isomorphism φ. Then there exists an invertible central element c of $\mathscr{U}(G)$ such that $\mathscr{F}' = (c\otimes c)\mathscr{F}\hat\Delta(c)^{-1}$. When φ is a $*$-isomorphism and both Drinfeld twists are unitary, then c can also be chosen to be unitary.*

Proof. — To simplify the notation we shall omit φ in the computations, so we identify $\mathscr{U}(G_q)$ and $\mathscr{U}(G)$ as algebras. Set $\mathscr{E} = \mathscr{F}'\mathscr{F}^{-1}$. Then

$$\begin{aligned}(\iota\otimes\hat\Delta)(\mathscr{F}^{-1})(1\otimes\mathscr{F}^{-1})(\mathscr{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{F})&\\ = (\iota\otimes\hat\Delta)(\mathscr{F}^{-1}\mathscr{E}^{-1})(1\otimes\mathscr{F}^{-1}\mathscr{E}^{-1})(\mathscr{E}\mathscr{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathscr{E}\mathscr{F}).&\end{aligned}$$

Multiplying by $(1\otimes\mathscr{F})(\iota\otimes\hat\Delta)(\mathscr{F})$ on the left and by $(\hat\Delta\otimes\iota)(\mathscr{F}^{-1})(\mathscr{F}^{-1}\otimes 1)$ on the right, and using that $\mathscr{F}\hat\Delta(\cdot)\mathscr{F}^{-1} = \hat\Delta_q$, we get

$$1 = (\iota\otimes\hat\Delta_q)(\mathscr{E}^{-1})(1\otimes\mathscr{E}^{-1})(\mathscr{E}\otimes 1)(\hat\Delta_q\otimes\iota)(\mathscr{E}).$$

Therefore $\mathscr{E}$ is a dual 2-cocycle on G_q. Since

$$\mathscr{E}\hat\Delta_q(\cdot)\mathscr{E}^{-1} = \mathscr{E}\mathscr{F}\hat\Delta(\cdot)\mathscr{F}^{-1}\mathscr{E}^{-1} = \mathscr{F}'\hat\Delta(\cdot)\mathscr{F}'^{-1} = \hat\Delta_q,$$

the cocycle $\mathscr{E}\in\mathscr{U}(G_q\times G_q)$ is invariant, and since

$$\mathscr{E}_{21}\mathscr{R}_q\mathscr{E}^{-1} = \mathscr{E}_{21}\mathscr{F}_{21}q^t\mathscr{F}^{-1}\mathscr{E}^{-1} = \mathscr{F}'_{21}q^t\mathscr{F}'^{-1} = \mathscr{R}_q,$$

it is symmetric. By Corollary 3.4.12 there exists a central element c of $\mathscr{U}(G_q) = \mathscr{U}(G)$ that is unitary if $\mathscr{E}$ is unitary, such that

$$\mathscr{E} = (c\otimes c)\hat\Delta_q(c)^{-1},$$

so that $\mathscr{F}' = (c\otimes c)\hat\Delta_q(c^{-1})\mathscr{F} = (c\otimes c)\mathscr{F}\hat\Delta(c^{-1})$. □

Remark 4.3.7. — By Corollary 3.4.2, if $\mathfrak{g}$ is simple and $\mathfrak{g}\not\cong\mathfrak{so}_{4n}(\mathbb{C})$, then any invariant dual 2-cocycle on G_q is a coboundary. The above proof then implies that for any such $\mathfrak{g}$ the condition $(\varphi\otimes\varphi)(\mathscr{R}_q) = \mathscr{F}_{21}q^t\mathscr{F}^{-1}$ follows from the other three conditions on Drinfeld twists. It follows that any monoidal equivalence between $\mathscr{D}(\mathfrak{g},\hbar)$ and $\mathscr{C}_q(\mathfrak{g})$ is automatically braided.

We can now prove the following strengthening of Proposition 4.3.5.

Theorem 4.3.8. — *Assume $\{\varphi^q\colon \mathscr{U}(G_q)\to\mathscr{U}(G)\}_q$ is a continuous family of $*$-isomorphisms such that $\varphi^1 = \iota$. Then there exists a continuous family of unitary Drinfeld twists $\mathscr{F}^q$ corresponding to φ^q such that $\mathscr{F}^1 = 1$.*

Furthermore, if $\{\psi^q\colon \mathscr{U}(G_q)\to\mathscr{U}(G)\}_{q>0}$ is another continuous family of $*$-isomorphisms such that $\psi^1 = \iota$, and $\{\mathscr{E}^q\}_{q>0}$ is a corresponding continuous family of unitary Drinfeld twists *with* $\mathscr{E}^1 = 1$, *then there exists a unique continuous family of unitary elements* $u^q \in \mathscr{U}(G)$ *such that*

$$u^1 = 1,\ \psi^q = u^q\varphi^q(\cdot)u^{q*}\ \ \textit{and}\ \mathscr{E}^q = (u^q \otimes u^q)\mathscr{F}^q\hat{\Delta}(u^q)^*\ \ \textit{for all}\ q > 0.$$

Proof. — To prove the existence of $\mathscr{F}_q$, consider the set Ω of pairs $(q, \mathscr{F})$, where $q > 0$ and $\mathscr{F}$ is a unitary Drinfeld twist for φ^q. Let $p\colon \Omega \to \mathbb{R}_+^*$ be the projection onto the first coordinate. The compact abelian group of elements of the form $(c \otimes c)\hat{\Delta}(c)^*$, where c is a unitary element in the center of $\mathscr{U}(G)$, acts freely by multiplication on the right on the locally compact space Ω, and by Theorem 4.3.6 this action is transitive on each fiber of the map p. Therefore if this group were a compact Lie group, then by a theorem of Gleason [**35**], $p\colon \Omega \to \mathbb{R}_+^*$ would be a fiber bundle, hence p would have a continuous section. Since the group of elements of the form $(c \otimes c)\hat{\Delta}(c)^*$ is not a Lie group, we cannot apply Gleason's theorem directly and will proceed as follows.

Choose an increasing sequence of finite subsets $P_n \subset P_+$ such that $P_1 = \{0\}$ and $\cup_n P_n = P_+$. For every $q > 0$ we will construct a sequence of unitary Drinfeld twists $\mathscr{F}_n^q$ such that $\mathscr{F}_n^1 = 1$, the map $q \mapsto (\pi_\lambda \otimes \pi_\nu)(\mathscr{F}_n^q)$ is continuous for all $\lambda, \nu \in P_n$ and $n \geq 1$, and $(\pi_\lambda \otimes \pi_\nu)(\mathscr{F}_{n+1}^q) = (\pi_\lambda \otimes \pi_\nu)(\mathscr{F}_n^q)$ for all $\lambda, \nu \in P_n$ and $n \geq 1$. Then, for every $q > 0$, the sequence $\{\mathscr{F}_n^q\}_n$ converges to a unitary Drinfeld twist $\mathscr{F}^q$ with the required properties.

For $n = 1$ and $q \neq 1$ we take $\mathscr{F}_1^q$ to be any unitary Drinfeld twist, and we take $\mathscr{F}_1^1 = 1$.

Assume the Drinfeld twists $\mathscr{F}_n^q$ are already constructed for some n. Denote by Ω_{n+1} the set of pairs (q, W), where $W = (W_{\lambda,\mu})_{\lambda,\mu}$ is a unitary element in

$$\prod_{(\lambda,\mu)\in P_{n+1}\times P_{n+1}\setminus P_n\times P_n} B(V_\lambda \otimes V_\mu)$$

such that there exists a unitary Drinfeld twist $\mathscr{F}$ for φ^q satisfying

$$(\pi_\lambda \otimes \pi_\mu)(\mathscr{F}) = W_{\lambda,\mu}\ \text{ for all }\ (\lambda,\mu) \in P_{n+1} \times P_{n+1} \setminus P_n \times P_n,$$
$$(\pi_\lambda \otimes \pi_\mu)(\mathscr{F}) = (\pi_\lambda \otimes \pi_\mu)(\mathscr{F}_n^q)\ \text{ for all }\ \lambda, \mu \in P_n.$$

Let $p_{n+1}\colon \Omega_{n+1} \to \mathbb{R}_+^*$ be the projection onto the first coordinate. The set Ω_{n+1} is a closed subset of

$$\mathbb{R}_+^* \times \prod_{(\lambda,\mu)\in P_{n+1}\times P_{n+1}\setminus P_n\times P_n} U(V_\lambda \otimes V_\mu).$$

For every $q > 0$ the fiber $p_{n+1}^{-1}(q)$ is nonempty, since it contains the element

$$((\pi_\lambda \otimes \pi_\mu)(\mathscr{F}_n^q))_{\lambda,\mu}.$$

Let S_{n+1} be the set of weights $\lambda \in P_+$ such that either $\lambda \in P_{n+1}$ or V_λ is equivalent to a subrepresentation of $V_\mu \otimes V_\eta$ for some $\mu, \eta \in P_{n+1}$. Put $K_{n+1} = \prod_{\lambda\in S_{n+1}} \mathbb{T}$. We have a homomorphism

$$\rho_{n+1}\colon K_{n+1}\to\prod_{(\lambda,\mu)\in P_{n+1}\times P_{n+1}\setminus P_n\times P_n} U(V_\lambda\otimes V_\mu)$$

such that $\rho_{n+1}(c)$ acts on the isotypic component of $V_\mu\otimes V_\eta$ of type V_λ as multiplication by $c_\mu c_\eta\bar c_\lambda$. We also have a similar homomorphism θ_{n+1} from K_{n+1} into the unitary group of $\prod_{\lambda,\mu\in P_n} B(V_\lambda\otimes V_\mu)$.

The group $\ker\theta_{n+1}\subset K_{n+1}$ acts on Ω_{n+1} by multiplication by $\rho_{n+1}(c)$ on the right. On every fiber of p_{n+1} this action is transitive, and the stabilizer of every point is $\ker\rho_{n+1}\cap\ker\theta_{n+1}$. Since $\ker\theta_{n+1}$ is a compact Lie group, by Gleason's theorem we conclude that $p_{n+1}\colon\Omega_{n+1}\to\mathbb{R}^*_+$ is a fiber bundle, hence it is a trivial bundle. Choosing a continuous section of this bundle, by definition of Ω_{n+1} we conclude that there exist unitary Drinfeld twists $\mathscr{E}^q$ such that the map $q\mapsto(\pi_\lambda\otimes\pi_\mu)(\mathscr{E}^q)$ is continuous for all $\lambda,\mu\in P_{n+1}$ and $(\pi_\lambda\otimes\pi_\mu)(\mathscr{E}^q)=(\pi_\lambda\otimes\pi_\mu)(\mathscr{F}^q_n)$ for all $\lambda,\mu\in P_n$. There exists a unitary central element c in $\mathscr{U}(G)$ such that $\mathscr{E}^1=(c^*\otimes c^*)\hat\Delta(c)$. We can then set $\mathscr{F}^q_{n+1}=\mathscr{E}^q(c\otimes c)\hat\Delta(c)^*$. This finishes the proof of the induction step.

Assume now that $\{\psi^q\colon\mathscr{U}(G_q)\to\mathscr{U}(G)\}_{q>0}$ is another continuous family of $*$-isomorphisms such that $\psi^1=\iota$, and $\{\mathscr{E}^q\}_{q>0}$ is a corresponding continuous family of unitary Drinfeld twists with $\mathscr{E}^1=1$. For every $\lambda\in P_+$, let $\varphi^q_\lambda,\psi^q_\lambda\colon B(V^q_\lambda)\to B(V_\lambda)$ be the isomorphisms defined by φ^q and ψ^q. The set of unitaries $v\in B(V_\lambda)$ such that $\psi^q_\lambda=v\varphi^q_\lambda(\cdot)v^*$ forms a circle bundle over $\mathbb{R}^*_+$, so it has a continuous section v^q_λ. Since $\psi^1_\lambda=\varphi^1_\lambda=\iota$, we may assume that $v^1_\lambda=1$. The unitaries v^q_λ define a continuous family of unitaries $v^q\in\mathscr{U}(G)$.

For every $q>0$, the element $(v^q\otimes v^q)\mathscr{F}^q\hat\Delta(v^q)^*$ is a unitary Drinfeld twist for ψ^q. Hence, for every q, there exists a unitary central element $c\in\mathscr{U}(G)$ such that

$$\mathscr{E}^q=(v^q\otimes v^q)\mathscr{F}^q\hat\Delta(v^q)^*(c\otimes c)\hat\Delta(c)^*. \tag{4.3.2}$$

Furthermore, the element c is defined up to a group-like unitary element in the center of $\mathscr{U}(G)$, that is, by Theorem 3.2.1, up to an element of the center $Z(G)$ of G. Therefore, applying once again Gleason's theorem (which in this case is quite obvious as $Z(G)$ is finite), we see that the set of pairs (q,c) with c satisfying (4.3.2) is a principal $Z(G)$-bundle over $\mathbb{R}^*_+$, hence it has a continuous section $q\mapsto(q,c^q)$. The element c^1 is group-like, so replacing c^q by $c^q{c^1}^*$ we may assume that $c^1=1$. Letting $u^q=c^qv^q$, we get the required continuous family of unitary elements.

Finally, if $\tilde u^q$ is another continuous family of unitary elements with the same properties, then $c^q=\tilde u^q u^{q*}$ is a unitary central group-like element in $\mathscr{U}(G)$, hence $c^q\in Z(G)$. Since c^q depends continuously on q, $Z(G)$ is finite and $c^1=1$, we conclude that $c^q=1$ for all q. □

We finish the section by explaining Drinfeld's initial motivation for defining $\mathscr{D}(\mathfrak{g},\hbar)$ and proving the equivalence of categories. Given a strict braided C*-tensor category $\mathscr{C}$ and an object V in $\mathscr{C}$, we can define a representation $\pi\colon B_n \to \mathrm{End}(V^{\otimes n})$ of the braid group B_n by $\pi(g_i) = \sigma_{i,i+1}$ for $1 \le i \le n-1$. Theorem 2.6.5(i) shows that the braid relations are indeed satisfied. More generally, even if the category is nonstrict, it is equivalent to a strict one, so we still get a representation of B_n on an object $V^{\otimes n}$ that is defined as the n-th tensor power of V with some fixed arrangement of brackets; for example, we can take

$$V^{\otimes n} = (\dots((V \otimes V) \otimes V)\dots) \otimes V.$$

The following result is a version of the famous Kohno-Drinfeld theorem.

Theorem 4.3.9. — *Assume $\hbar \in i\mathbb{R}$ and $q = e^{\pi i\hbar}$. Let V be a finite dimensional $\mathfrak{g}$-module and V^q be the corresponding $U_q\mathfrak{g}$-module, so the multiplicity of V^q_λ in V^q is the same as the multiplicity of V_λ in V for every $\lambda \in P_+$. Then the representation of B_n on $V^{\otimes n}$ defined by monodromy of the KZ_n equations with the parameter $\hbar$ is equivalent to the representation of B_n on $(V^q)^{\otimes n}$ defined by braiding on $\mathscr{C}_q(\mathfrak{g})$.*

Proof. — First one has to show that the representation of B_n on $V^{\otimes n}$ defined by monodromy is equivalent to the representation defined by braiding on $\mathscr{D}(\mathfrak{g},\hbar)$ corresponding to some fixed arrangement of brackets on $V^{\otimes n}$. For $n = 2$ this is easy, while for $n = 3$ this follows from our discussion in Section 4.1, Indeed, we showed there that the representation defined by monodromy with respect to some base point is equivalent to

$$g_1 \mapsto \Sigma_{12}e^{\pi i\hbar t_{12}},\ \ g_2 \mapsto \Phi(\hbar t_{12}, \hbar t_{23})^{-1}\Sigma_{23}e^{\pi i\hbar t_{23}}\Phi(\hbar t_{12}, \hbar t_{23}),$$

which is exactly the representation defined by braiding on $(V \otimes V) \otimes V$. For $n > 3$ this requires a bit more thorough discussion of monodromy of the KZ-equations, which we omit, see e.g., [**30**].

Consider now a braided monoidal equivalence $F\colon \mathscr{D}(\mathfrak{g},\hbar) \to \mathscr{C}_q(\mathfrak{g})$ as in Theorem 4.2.1. We can take $V^q = F(V)$. Let $\pi^\hbar$ be the representation of B_n on $V^{\otimes n}$ defined by braiding on $\mathscr{D}(\mathfrak{g},\hbar)$, and π^q be the representation of B_n on $F(V)^{\otimes n}$ defined by braiding on $\mathscr{C}_q(\mathfrak{g})$. The tensor functor gives us an isomorphism $F(V^{\otimes n}) \to F(V)^{\otimes n}$, which intertwines $F\pi^\hbar$ with π^q. On the other hand, as we have already used in the proof of Theorem 4.3.1, there exists a natural isomorphism η of the forgetful functor on $\mathscr{D}(\mathfrak{g},\hbar)$ and the functor F. Then by naturality $\eta_{V^{\otimes n}}\colon V^{\otimes n} \to F(V^{\otimes n})$ intertwines $\pi^\hbar$ with $F\pi^\hbar$. Hence the representations $\pi^\hbar$ and π^q are equivalent.

Somewhat more explicitly this equivalence can be described as follows. To simplify the notation consider the case $n = 3$. Fix an isomorphism $\varphi\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ and a Drinfeld twist $\mathscr{F}$. This gives us a braided monoidal equivalence $F\colon \mathscr{D}(\mathfrak{g},\hbar) \to \mathscr{C}_q(\mathfrak{g})$, with $F(V) = V$ considered as a $U_q\mathfrak{g}$-module using φ and $F_2 = \mathscr{F}^{-1}$. Then

$$(\mathscr{F} \otimes \iota)(\hat{\Delta} \otimes \iota)(\mathscr{F}) : V^{\otimes 3} \to V^{\otimes 3}$$

intertwines the representation $\pi^{\hbar}$ of B_3 on $(V \otimes V) \otimes V$ with the representation π^q on $F(V)^{\otimes 3} = V^{\otimes 3}$. □

References. — **[27]**, **[30]**, **[52]**, **[66]**, **[68]**, **[70]**, **[67]**, **[72]**.

4.4. NORMALIZATION OF DRINFELD TWISTS

In this last section we will study the behavior of Drinfeld twists under the action of the antipode, and will show that there is a class of Drinfeld twists for which this action has a particularly nice form.

It will be convenient to choose an isomorphism $\varphi\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ satisfying additional properties. Recall that we denote by $\hat{R}_q$ the unitary antipode on $\mathscr{U}(G_q)$. For $q = 1$ it coincides with the antipode $\hat{S}$.

Lemma 4.4.1. — *For every $q > 0$ there exists a $*$-isomorphism $\varphi\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ that extends the canonical identification of the centers and satisfies the property $\varphi\hat{R}_q = \hat{S}\varphi$. Furthermore, such isomorphisms can be chosen to be continuous in q and such that they are the identity maps on the maximal torus T.*

Proof. — If we identify $\mathscr{U}(G)$ with $\prod_{\lambda\in P_+} B(V_\lambda)$ and $\mathscr{U}(G_q)$ with $\prod_{\lambda\in P_+} B(V_\lambda^q)$, then choosing φ is the same as choosing $*$-isomorphisms $\varphi_\lambda\colon B(V_\lambda^q) \to B(V_\lambda)$. Let $\bar{\lambda}$ be the highest weight of $\bar{V}_\lambda^q$; we know that $\bar{\lambda} = -w_0\lambda$, but this will not be important. The unitary antipode maps $B(V_\lambda^q)$ onto $B(V_{\bar{\lambda}}^q)$. For every representative λ of a set $\{\lambda, \bar{\lambda}\}$ with $\bar{\lambda} \neq \lambda$, we can take φ_λ arbitrary and then let $\varphi_{\bar{\lambda}} = \hat{S}\varphi_\lambda\hat{R}_q$.

Assume now that $\bar{\lambda} = \lambda$, so $\bar{V}_\lambda^q \cong V_\lambda^q$. This means that there exists an anti-linear isometry $J_q\colon V_\lambda^q \to V_\lambda^q$ such that $J_q\omega = \hat{R}_q(\omega)^* J_q$ for $\omega \in \mathscr{U}(G_q)$. To construct φ_λ we have to find a unitary $V_\lambda^q \to V_\lambda$ intertwining J_q with $J = J_1$. Since $\hat{R}_q^2 = \iota$, the operator J_q^2 is a scalar, say $J_q^2 = c_q 1$ for some $c_q \in \mathbb{T}$, and then

$$c_q J_q = J_q^3 = J_q c_q 1 = \bar{c}_q J_q,$$

whence $c_q = \pm 1$. Furthermore, the operator J_q is uniquely determined up to a phase factor, while the sign of J_q^2 is independent of any choices.

Claim 1. The sign of J_q^2 is the same for all $q > 0$.

We will show that the sign depends continuously on q. Multiplying J_q by a phase factor we may assume that J_q maps the highest weight vector ξ_λ^q onto a fixed lowest weight vector ζ_λ^q of norm one. There exist indices $i_1, \dots, i_n$ such that the vector $\tilde{\zeta}_\lambda^q =$

$F^q_{i_1}\dots F^q_{i_n}\xi^q_\lambda$ is nonzero. Then we can take $\zeta^q_\lambda = \tilde\zeta^q_\lambda/\|\tilde\zeta^q_\lambda\|$. Furthermore, as we have already used in the proof of Lemma 4.3.4, the scalar products of vectors of the form $F^q_{j_1}\dots F^q_{j_m}\xi^q_\lambda$ depend continuously on q. Hence the same indices $i_1,\dots,i_n$ can be used to define a lowest weight vector for all values of the deformation parameter close to q. We have

$$\begin{aligned}(J_q^2\xi^q_\lambda,\xi^q_\lambda) &= \|\tilde\zeta^q_\lambda\|^{-1}(J_qF^q_{i_1}\dots F^q_{i_n}\xi^q_\lambda,\xi^q_\lambda)\\ &= \|\tilde\zeta^q_\lambda\|^{-1}(\hat R_q(F^q_{i_1})^*\dots \hat R_q(F^q_{i_n})^*J_q\xi^q_\lambda,\xi^q_\lambda)\\ &= \|\tilde\zeta^q_\lambda\|^{-2}(F^q_{i_1}\dots F^q_{i_n}\xi^q_\lambda,\hat R_q(F^q_{i_n})\dots \hat R_q(F^q_{i_1})\xi^q_\lambda).\end{aligned}$$

As $\hat R_q(F_i) = -q_i^{-1}F_iK_i$ by (2.4.2), we see that $(J_q^2\xi^q_\lambda,\xi^q_\lambda)$ depends continuously on q.

The first part of the lemma follows now from the following elementary result.

Claim 2. For any finite dimensional Hilbert space H, the unitary conjugacy class of an anti-linear isometry J on H such that $J^2=\pm1$ is completely determined by the sign of J^2.

Indeed, if $J^2=1$, then the space of vectors invariant under J forms a real form of the Hilbert space H, and any two real forms are unitarily conjugate. If $J^2=-1$, then for any vector ξ we have $J\xi\perp\xi$, since

$$(\xi,J\xi) = (JJ\xi,J\xi) = -(\xi,J\xi).$$

This implies that there exists an orthonormal basis in H of the form

$$e_1, Je_1,\dots,e_m,Je_m.$$

Any two such J are clearly unitarily conjugate.

In order to prove the second part, recall that by Lemma 3.2.4 there already exists a continuous family of isomorphisms $\varphi^q=(\varphi^q_\lambda)\colon \mathscr U(G_q)\to\mathscr U(G)$ that are the identity maps on T. We modify φ^q_λ as follows. For every representative λ of a set $\{\lambda,\bar\lambda\}$ with $\bar\lambda\ne\lambda$, we keep φ^q_λ intact and let $\varphi^q_{\bar\lambda}=\hat S\varphi^q_\lambda\hat R_q$. In the case $\bar\lambda=\lambda$ recall from the proof of Lemma 3.2.4 that locally φ^q_λ are implemented by unitaries $u^q_\lambda\colon V^q_\lambda\to V_\lambda$ defined using orthonormal bases in $V^q_\lambda(\mu)$ that consist of linear combinations of vectors of the form $F^q_{j_1}\dots F^q_{j_m}\xi^q_\lambda$ with coefficients continuous in q. If J_q is chosen as in the proof of Claim 1 above, then the matrices of J_q in these bases depend continuously on q, so the anti-linear isometries $u^q_\lambda J_q u^{q*}_\lambda$ on V_λ are continuous in q. Note also that J_q maps $V^q_\lambda(\mu)$ onto $V^q_\lambda(-\mu)$. We now modify u^q_λ as follows. For every representative μ of a set $\{\mu,-\mu\}$ with $\mu\ne0$, we keep the restriction of u^q_λ to $V^q_\lambda(\mu)$ intact and let $u^q_\lambda=J^{-1}u^q_\lambda J_q$ on $V^q_\lambda(-\mu)$. Finally, it is clear from the proof of Claim 2 that we can choose a continuous family of unitaries w^q on $V_\lambda(0)$ such that $w^qu^q_\lambda J_qu^{q*}_\lambda w^{q*}=J$. We then replace the restriction of u^q_λ to $V^q_\lambda(0)$ by $w^qu^q_\lambda$. □

From now on fix $q>0$ and a $*$-isomorphism $\varphi\colon\mathscr U(G_q)\to\mathscr U(G)$ such that it extends the canonical identification of the centers, acts as the identity map on the maximal torus T, and satisfies $\varphi\hat{R}_q = \hat{S}\varphi$. By (2.4.1) it follows that

$$\varphi\hat{S}_q = (\operatorname{Ad} q^{-h_\rho})\hat{S}\varphi.$$

Denote by γ_q the central element in $\mathscr{U}(G)$ that acts on V_λ as multiplication by

$$\left(\frac{\dim V_\lambda}{\dim_q V_\lambda^q}\right)^{1/2}.$$

Theorem 4.4.2. — *Let $\mathscr{F} \in \mathscr{U}(G \times G)$ be a unitary Drinfeld twist for G_q corresponding to the isomorphism φ. Then*

$$m(\iota \otimes \hat{S})(\mathscr{F}) = v\gamma_q q^{-h_\rho}$$

for some $\hat{S}$-invariant central unitary element $v \in \mathscr{U}(G)$, where $m\colon \mathscr{U}(G \times G) \to \mathscr{U}(G)$ is the multiplication map. In particular, there exists a unitary Drinfeld twist $\mathscr{F}$ such that $m(\iota \otimes \hat{S})(\mathscr{F}) = \gamma_q q^{-h_\rho}$.

We will divide the proof into several lemmas. We will use notation from Section 2.6 by working with elements of $\mathscr{U}(G^n)$ as if they were elementary tensors. Then, for example, the element $m(\iota \otimes \hat{S})(\mathscr{F})$ can be written as $\mathscr{F}_1\hat{S}(\mathscr{F}_2)$.

Let $\hbar \in i\mathbb{R}$ be such that $q = e^{\pi i \hbar}$.

Lemma 4.4.3. — *For the element $\Phi = \Phi(\hbar t_{12}, \hbar t_{23})$ we have $\Phi_1\hat{S}(\Phi_2)\Phi_3 = \gamma_q^2$.*

Proof. — For $\lambda \in P_+$, the element $\Phi_1\hat{S}(\Phi_2)\Phi_3$ acts on V_λ as the composition

$$V_\lambda \xrightarrow{\bar{r}\otimes\iota} (V_\lambda \otimes \bar{V}_\lambda) \otimes V_\lambda \xrightarrow{\Phi} V_\lambda \otimes (\bar{V}_\lambda \otimes V_\lambda) \xrightarrow{\iota\otimes r^*} V_\lambda,$$

where $(r, \bar{r})$ is the canonical solution of the conjugate equations for the space V_λ defined in Example 2.2.2. At the same time, since $\mathscr{D}(\mathfrak{g}, \hbar)$ is unitarily monoidally equivalent to $\mathscr{C}_q(\mathfrak{g})$, there exists a solution $(R, \bar{R})$ of the conjugate equations for $(V_\lambda, \bar{V}_\lambda)$ such that $\|R\| = \|\bar{R}\| = (\dim_q V_\lambda^q)^{1/2}$. Then by definition the above composition with $(r, \bar{r})$ replaced by $(R, \bar{R})$ is the identity map. Since the space of morphisms $\mathbb{1} \to V_\lambda \otimes \bar{V}_\lambda$ is one-dimensional, the morphisms R and $\bar{R}$ coincide with r and $\bar{r}$ up to scalar factors. Since $\|r\| = \|\bar{r}\| = (\dim V_\lambda)^{1/2}$, it follows that $\Phi_1\hat{S}(\Phi_2)\Phi_3$ acts on V_λ as a scalar $c_q(\lambda)$ of modulus $\dfrac{\dim V_\lambda}{\dim_q V_\lambda^q}$. The scalars $c_q(\lambda)$ depend continuously on q.

As we remarked at the end of the proof of Theorem 4.1.4, we have $\Phi^{-1} = \Phi_{321}$. Since Φ is unitary, it follows that the element $\Phi_1\hat{S}(\Phi_2)\Phi_3$ is self-adjoint. Hence $c_q(\lambda)$ is real, and therefore it must be equal to $\dfrac{\dim V_\lambda}{\dim_q V_\lambda^q}$. □

Put $u = \mathscr{F}_1\hat{S}(\mathscr{F}_2)$.

Lemma 4.4.4. — *We have $u^{-1} = \gamma_q^{-2}\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*$.*

Proof. — Apply $m^{(2)}(\iota\otimes\hat{S}\otimes\iota)$, where $m^{(2)}=m(m\otimes\iota)$, to the identity

$$\Phi=(\iota\otimes\hat{\Delta})(\mathscr{F}^*)(1\otimes\mathscr{F}^*)(\mathscr{F}\otimes 1)(\hat{\Delta}\otimes\iota)(\mathscr{F}). \tag{4.4.1}$$

By the previous lemma, on the left hand side we get γ_q^2. In order to compute the right hand side observe that for every element $\omega\in\mathscr{U}(G^3)$ we have

$$\begin{aligned}m^{(2)}(\iota\otimes\hat{S}\otimes\iota)(\omega(\hat{\Delta}\otimes\iota)(\mathscr{F}))&=\omega_1\mathscr{F}_{1,(1)}\hat{S}(\mathscr{F}_{1,(2)})\hat{S}(\omega_2)\omega_3\mathscr{F}_2\\&=\omega_1\hat{S}(\omega_2)\omega_3(\hat{\varepsilon}\otimes\iota)(\mathscr{F})=\omega_1\hat{S}(\omega_2)\omega_3.\end{aligned}$$

Similarly,

$$m^{(2)}(\iota\otimes\hat{S}\otimes\iota)((\iota\otimes\hat{\Delta})(\mathscr{F}^*)\omega)=m^{(2)}(\iota\otimes\hat{S}\otimes\iota)(\omega).$$

Using these identities, by applying $m^{(2)}(\iota\otimes\hat{S}\otimes\iota)$ to the right hand side of (4.4.1) we get

$$m^{(2)}(\iota\otimes\hat{S}\otimes\iota)((1\otimes\mathscr{F}^*)(\mathscr{F}\otimes 1))=\mathscr{F}_1\hat{S}(\mathscr{F}_2)\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*,$$

so $\gamma_q^2=u\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*$. □

Lemma 4.4.5. — *We have $\varphi\hat{S}_q=(\operatorname{Ad}u)\hat{S}\varphi$. Therefore the element uq^{h_ρ} is central.*

Proof. — We claim that the map $\hat{S}_u=(\operatorname{Ad}u)\hat{S}$ is an antipode for the comultiplication $\hat{\Delta}_{\mathscr{F}}=(\operatorname{Ad}\mathscr{F})\hat{\Delta}$ in the sense that

$$m(\hat{S}_u\otimes\iota)\hat{\Delta}_{\mathscr{F}}=\hat{\varepsilon}(\cdot)1=m(\iota\otimes\hat{S}_u)\hat{\Delta}_{\mathscr{F}}.$$

We will only check the first identity. We have

$$\begin{aligned}m(\hat{S}_u\otimes\iota)\hat{\Delta}_{\mathscr{F}}(\omega)&=m(u\hat{S}(\cdot)u^{-1}\otimes\iota)(\mathscr{F}_1\omega_{(1)}\mathscr{F}_1^*\otimes\mathscr{F}_2\omega_{(2)}\mathscr{F}_2^*)\\&=u\hat{S}(\mathscr{F}_1^*)\hat{S}(\omega_{(1)})\hat{S}(\mathscr{F}_1)u^{-1}\mathscr{F}_2\omega_{(2)}\mathscr{F}_2^*.\end{aligned}$$

By the previous lemma,

$$\hat{S}(\mathscr{F}_1)u^{-1}\mathscr{F}_2=\gamma_q^{-2}\hat{S}(\mathscr{F}_1)\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*\mathscr{F}_2=\gamma_q^{-2}m(\hat{S}\otimes\iota)(\mathscr{F}^*\mathscr{F})=\gamma_q^{-2}.$$

Therefore

$$m(\hat{S}_u\otimes\iota)\hat{\Delta}_{\mathscr{F}}(\omega)=\gamma_q^{-2}u\hat{S}(\mathscr{F}_1^*)\hat{S}(\omega_{(1)})\omega_{(2)}\mathscr{F}_2^*=\hat{\varepsilon}(\omega)\gamma_q^{-2}u\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*=\hat{\varepsilon}(\omega)1.$$

On the other hand, $\hat{S}'=\varphi\hat{S}_q\varphi^{-1}$ is also an antipode for $\Delta_{\mathscr{F}}$. The standard proof of the uniqueness of the antipode shows that $\hat{S}'=\hat{S}_u$:

$$\begin{aligned}\hat{S}'(\omega)&=\hat{S}'(\iota\otimes\hat{\varepsilon})\hat{\Delta}_{\mathscr{F}}(\omega)=m(\hat{S}'\otimes m(\iota\otimes\hat{S}_u)\hat{\Delta}_{\mathscr{F}})\hat{\Delta}_{\mathscr{F}}(\omega)\\&=m^{(2)}(\hat{S}'\otimes\iota\otimes\hat{S}_u)\hat{\Delta}^{(2)}_{\mathscr{F}}(\omega)=\hat{S}_u(\omega).\end{aligned}$$

The last statement in the formulation follows from $\varphi\hat{S}_q=(\operatorname{Ad}q^{-h_\rho})\hat{S}\varphi$. □

Lemma 4.4.6. — *We have $\hat{S}(u)u^*=\gamma_q^2$ and $uu^*=\gamma_q^2q^{-h_{2\rho}}$.*

Proof. — The first identity easily follows from Lemma 4.4.4:

$$u^* = (\mathscr{F}_1\hat{S}(\mathscr{F}_2))^* = \hat{S}(\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*) = \hat{S}(\gamma_q^2u^{-1}) = \gamma_q^2\hat{S}(u)^{-1}.$$

For the second one, we have

$$uu^* = \mathscr{F}_1\hat{S}(\mathscr{F}_2)\hat{S}(\mathscr{F}_2^*)\mathscr{F}_1^* = m^{(2)}(\iota\otimes\hat{S}\otimes\iota)((1\otimes\mathscr{F}_{21}^*)(\mathscr{F}\otimes 1)).$$

Put $\mathscr{R} = (\varphi\otimes\varphi)(\mathscr{R}_q) = \mathscr{F}_{21}q^t\mathscr{F}^*$. Then $\mathscr{F}_{21}^* = q^t\mathscr{F}^*\mathscr{R}^{-1}$, and we get

$$\begin{aligned} uu^* &= m^{(2)}(\iota\otimes\hat{S}\otimes\iota)((1\otimes q^t\mathscr{F}^*\mathscr{R}^{-1})(\mathscr{F}\otimes 1)) \\ &= \mathscr{F}_1\hat{S}(\mathscr{F}_2)\hat{S}((\mathscr{R}^{-1})_1)\hat{S}(\mathscr{F}_1^*)\hat{S}((q^t)_1)(q^t)_2\mathscr{F}_2^*(\mathscr{R}^{-1})_2. \end{aligned}$$

From the identity $t = \frac{1}{2}(\hat{\Delta}(C) - 1\otimes C - C\otimes 1)$, where C is the Casimir operator, we get

$$\hat{S}((q^t)_1)(q^t)_2 = q^{-C}.$$

It follows that

$$uu^* = q^{-C}\mathscr{F}_1\hat{S}(\mathscr{F}_2)\hat{S}((\mathscr{R}^{-1})_1)\hat{S}(\mathscr{F}_1^*)\mathscr{F}_2^*(\mathscr{R}^{-1})_2 = q^{-C}\gamma_q^2u\hat{S}((\mathscr{R}^{-1})_1)u^{-1}(\mathscr{R}^{-1})_2.$$

Now recall from Example 2.6.12 that the twist θ for G_q equals q^{-C_q}, while $\theta^{-1} = \hat{S}_q((\mathscr{R}_q^{-1})_1)(\mathscr{R}_q^{-1})_2q^{h_{2\rho}}$ by (2.6.5). This shows that $\hat{S}_q((\mathscr{R}_q^{-1})_1)(\mathscr{R}_q^{-1})_2 = q^{C_q}q^{-h_{2\rho}}$, which means that

$$u\hat{S}((\mathscr{R}^{-1})_1)u^{-1}(\mathscr{R}^{-1})_2 = q^Cq^{-h_{2\rho}}.$$

Therefore $uu^* = q^{-C}\gamma_q^2q^Cq^{-h_{2\rho}} = \gamma_q^2q^{-h_{2\rho}}$. □

Proof of Theorem 4.4.2. — Since $uu^* = \gamma_q^2q^{-h_{2\rho}}$ by the previous lemma, we have by polar decomposition that $u = \gamma_qq^{-h_\rho}v$ for a unitary v. Since uq^{h_ρ} is central by Lemma 4.4.5, the element v must be central. From the identity $\hat{S}(u)u^* = \gamma_q^2$ we then get $\hat{S}(v)v^* = 1$, so v is $\hat{S}$-invariant.

If $v \neq 1$, consider the Drinfeld twist $\tilde{\mathscr{F}} = (w^*\otimes w^*)\mathscr{F}\hat{\Delta}(w)$, where w is a central unitary element such that $\hat{\varepsilon}(w) = 1$. Then

$$\tilde{\mathscr{F}}_1\hat{S}(\tilde{\mathscr{F}}_2) = w^*\mathscr{F}_1\hat{S}(\mathscr{F}_2)\hat{S}(w)^* = w^*\hat{S}(w)^*v\gamma_qq^{-h_\rho}.$$

Therefore to prove the second part of the theorem it suffices to show that for every central unitary $\hat{S}$-invariant element v such that $\hat{\varepsilon}(v) = 1$ there exists a central unitary element w such that $v = w\hat{S}(w)$ and $\hat{\varepsilon}(w) = 1$. This is straightforward, we can even choose w to be $\hat{S}$-invariant. □

For a unitary Drinfeld twist $\mathscr{F}$ such that $\mathscr{F}_1\hat{S}(\mathscr{F}_2) = \gamma_qq^{-h_\rho}$ consider the unitary element

$$\mathscr{E} = (\hat{S}\otimes\hat{S})(\mathscr{F}_{21})\mathscr{F}.$$

Theorem 4.4.7. — *The unitary $\mathscr{E}$ does not depend on the choice of φ and $\mathscr{F}$ with properties as described above. It satisfies the following properties:*

(i) $\mathscr{E} \in \mathscr{U}(G \times G)$ *is invariant, that is, it commutes with all elements of the form* $\hat{\Delta}(\omega)$*;*

(ii) $(\hat{\varepsilon} \otimes \iota)(\mathscr{E}) = (\iota \otimes \hat{\varepsilon})(\mathscr{E}) = 1$*;*

(iii) $\mathscr{E}_{21} = \mathscr{E}$*;*

(iv) $\Phi(\hbar t_{12}, \hbar t_{23}) = (\iota \otimes \hat{\Delta})(\mathscr{E}^*)(1 \otimes \mathscr{E}^*)\Phi(-\hbar t_{12}, -\hbar t_{23})(\mathscr{E} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathscr{E})$*;*

(v) $\mathscr{E}$ *is* $(\hat{S} \otimes \hat{S})$*-invariant;*

(vi) $m(\iota \otimes \hat{S})(\mathscr{E}) = 1$.

Proof. — We start by verifying property (i) for fixed φ and $\mathscr{F}$. Applying $\hat{S} \otimes \hat{S}$ to the identity

$$(\varphi \otimes \varphi)\hat{\Delta}_q = \mathscr{F}\hat{\Delta}\varphi(\cdot)\mathscr{F}^*$$

and using $\hat{S}\varphi = \varphi\hat{R}_q$, we get

$$(\varphi \otimes \varphi)\hat{\Delta}_q^{\mathrm{op}}\hat{R}_q = (\hat{S} \otimes \hat{S})(\mathscr{F}^*)\hat{\Delta}\varphi\hat{R}_q(\cdot)(\hat{S} \otimes \hat{S})(\mathscr{F}),$$

so that by applying the flip we obtain

$$(\varphi \otimes \varphi)\hat{\Delta}_q = (\hat{S} \otimes \hat{S})(\mathscr{F}_{21}^*)\hat{\Delta}\varphi(\cdot)(\hat{S} \otimes \hat{S})(\mathscr{F}_{21}).$$

It follows that the element $\mathscr{E} = (\hat{S} \otimes \hat{S})(\mathscr{F}_{21})\mathscr{F}$ is invariant.

Assume now that $(\tilde{\varphi}, \tilde{\mathscr{F}})$ is another pair with properties as prescribed before. By Theorem 4.3.6 and the discussion preceding it, there exists a unitary $v \in \mathscr{U}(G)$ such that $\tilde{\varphi} = (\operatorname{Ad} v)\varphi$, $\tilde{\mathscr{F}} = (v \otimes v)\mathscr{F}\hat{\Delta}(v)^*$ and $\hat{\varepsilon}(v) = 1$. Since $\tilde{\varphi}$ is assumed to be the identity map on the maximal torus, the element v commutes with q^{-h_ρ}. Then

$$m(\iota \otimes \hat{S})(\tilde{\mathscr{F}}) = v\, m(\iota \otimes \hat{S})(\mathscr{F})\hat{S}(v) = v\hat{S}(v)\gamma_q q^{-h_\rho}.$$

Hence $v\hat{S}(v) = 1$. Then

$$(\hat{S} \otimes \hat{S})(\tilde{\mathscr{F}}_{21})\tilde{\mathscr{F}} = \hat{\Delta}\hat{S}(v^*)(\hat{S} \otimes \hat{S})(\mathscr{F}_{21})(\hat{S}(v)v \otimes \hat{S}(v)v)\mathscr{F}\hat{\Delta}(v^*) = \hat{\Delta}\hat{S}(v^*)\mathscr{E}\hat{\Delta}(v^*).$$

The last expression is equal to $\mathscr{E}$, since $\mathscr{E}$ is invariant and $\hat{S}(v)v = 1$. Thus $\mathscr{E}$ is indeed independent of $(\varphi, \mathscr{F})$.

Property (ii) follows from the corresponding property of $\mathscr{F}$.

In order to prove property (iii) apply $\hat{S} \otimes \hat{S}$ to the identity

$$(\varphi \otimes \varphi)(\mathscr{R}_q) = \mathscr{F}_{21}q^t\mathscr{F}^*.$$

Using that $(\hat{R}_q \otimes \hat{R}_q)(\mathscr{R}_q) = \mathscr{R}_q$ by Theorem 2.6.6(iv), we get

$$(\varphi \otimes \varphi)(\mathscr{R}_q) = (\hat{S} \otimes \hat{S})(\mathscr{F}^*)q^t(\hat{S} \otimes \hat{S})(\mathscr{F}_{21}).$$

It follows that

$$\mathscr{E}_{21}q^t = q^t\mathscr{E}.$$

Since $\mathscr{E}$ is invariant and $t = \frac{1}{2}(\hat{\Delta}(C) - 1 \otimes C - C \otimes 1)$, the element $\mathscr{E}$ commutes with t. Hence $\mathscr{E}_{21} = \mathscr{E}$.

Turning to (iv), first note that if A and B are operators on a finite dimensional space V, and α is an anti-automorphism of $B(V)$, then $\alpha(\Phi(A,B))^{-1} = \Phi(-\alpha(A), -\alpha(B))$. This follows from the definition of Φ by observing that if G is an invertible solution of

$$G' = \left(\frac{A}{x} + \frac{B}{x-1}\right) G,$$

then for $\tilde{G} = \alpha(G)^{-1}$ we have

$$\begin{aligned}\tilde{G}' &= -\alpha(G)^{-1}\alpha(G')\alpha(G)^{-1}\\ &= -\alpha(G)^{-1}\alpha\left(\left(\frac{A}{x} + \frac{B}{x-1}\right) G\right)\alpha(G)^{-1} = -\left(\frac{\alpha(A)}{x} + \frac{\alpha(B)}{x-1}\right)\tilde{G}.\end{aligned}$$

It follows, by letting $\alpha = \hat{S}$, that

$$(\hat{S} \otimes \hat{S} \otimes \hat{S})(\Phi^*) = \Phi(-\hbar t_{12}, -\hbar t_{23}),$$

where $\Phi = \Phi(\hbar t_{12}, \hbar t_{23})$. At the same time

$$\Phi^* = \Phi_{321} = (\hat{\Delta} \otimes \iota)(\mathscr{F}_{21}^*)(\mathscr{F}_{21}^* \otimes 1)(1 \otimes \mathscr{F}_{21})(\iota \otimes \hat{\Delta})(\mathscr{F}_{21}),$$

so applying $\hat{S} \otimes \hat{S} \otimes \hat{S}$ and using $(\hat{S} \otimes \hat{S})(\mathscr{F}_{21}) = \mathscr{E}\mathscr{F}^*$ we get

$$\begin{aligned}(\hat{S} \otimes \hat{S} \otimes \hat{S})(\Phi^*) &= (\iota \otimes \hat{\Delta})(\mathscr{E}\mathscr{F}^*)(1 \otimes \mathscr{E}\mathscr{F}^*)(\mathscr{F}\mathscr{E}^* \otimes 1)(\hat{\Delta} \otimes \iota)(\mathscr{F}\mathscr{E}^*)\\ &= (\iota \otimes \hat{\Delta})(\mathscr{E})(1 \otimes \mathscr{E})\Phi(\mathscr{E}^* \otimes 1)(\hat{\Delta} \otimes \iota)(\mathscr{E}^*),\end{aligned}$$

where we again used invariance of $\mathscr{E}$. This proves (iv).

Property (v) is a consequence of (iii), since

$$(\hat{S} \otimes \hat{S})(\mathscr{E}) = (\hat{S} \otimes \hat{S})(\mathscr{F})\mathscr{F}_{21} = \mathscr{E}_{21}.$$

Finally, in order to prove property (vi) observe that since $\mathscr{E}$ is invariant, the element $c = \mathscr{E}_1\hat{S}(\mathscr{E}_2)$ is central. Indeed, if $\omega \in \mathscr{U}(G)$, then from $\mathscr{E}\hat{\Delta}(\omega) = \hat{\Delta}(\omega)\mathscr{E}$ we get $\hat{\varepsilon}(\omega)c = \omega_{(1)}c\hat{S}(\omega_{(2)})$, and writing $\omega_{(1)} \otimes \omega_{(2)} \otimes \omega_{(3)}$ for $\hat{\Delta}^{(2)}(\omega)$ we compute:

$$\omega c = \omega_{(1)}c\hat{\varepsilon}(\omega_{(2)}) = \omega_{(1)}c\hat{S}(\omega_{(2)})\omega_{(3)} = \hat{\varepsilon}(\omega_{(1)})c\omega_{(2)} = c\omega.$$

By applying $m(\iota \otimes \hat{S})$ to the identity $\mathscr{F} = (\hat{S} \otimes \hat{S})(\mathscr{F}_{21}^*)\mathscr{E}$ we now obtain

$$\gamma_q q^{-h_\rho} = \hat{S}(\mathscr{F}_2^*)c\mathscr{F}_1^* = c(\mathscr{F}_1\hat{S}(\mathscr{F}_2))^* = c\gamma_q q^{-h_\rho}.$$

Hence $c = 1$. □

For every fixed q properties (i)-(vi) in the above theorem do not completely determine $\mathscr{E}$, but the whole family of such elements for all $q > 0$ turns out to be uniquely determined by them. To formulate the result, observe that this family must be continuous. Indeed, by Lemma 4.4.1 there exists a continuous family of $*$-isomorphisms

$\varphi^q\colon \mathscr{U}(G_q) \to \mathscr{U}(G)$ such that they are the identity maps on the maximal torus and satisfy $\varphi^q \hat{R}_q = \hat{S}\varphi^q$. By Theorem 4.3.8 there exists a continuous family of unitary Drinfeld twists $\mathscr{F}^q$ corresponding to φ^q. By Theorem 4.4.2 we can modify $\mathscr{F}^q$ such that we get $m(\iota \otimes \hat{S})(\mathscr{F}^q) = \gamma_q q^{-h_\rho}$. Furthermore, from the proof of that theorem, which consists simply of taking square roots in the algebra of $\hat{S}$-invariant central elements, it is clear that we can arrange the modified family to remain continuous. Then the elements $(\hat{S} \otimes \hat{S})(\mathscr{F}^q_{21})\mathscr{F}^q$ depend continuously on q.

Theorem 4.4.8. — *There exists a unique continuous family $\{\mathscr{E}^q\}_{q>0}$ of unitary elements in $\mathscr{U}(G \times G)$ such that $\mathscr{E}^1 = 1$ and such that for every $q > 0$ the unitary $\mathscr{E}^q$ satisfies properties* (i)-(vi) *from Theorem 4.4.7 (with $\hbar \in i\mathbb{R}$ such that $q = e^{\pi i \hbar}$).*

Proof. — Let us first show that if $\mathscr{E}$ and $\tilde{\mathscr{E}}$ are two unitary elements satisfying properties (i)-(vi) for some $q > 0$, then there exists a central unitary element v such that $\tilde{\mathscr{E}} = (v \otimes v)\mathscr{E}\hat{\Delta}(v)^*$. Take a unitary Drinfeld twist $\mathscr{F}$ for G_q corresponding to some $*$-isomorphism $\mathscr{U}(G_q) \cong \mathscr{U}(G)$. Put $\tilde{\mathscr{F}} = \mathscr{F}\mathscr{E}^*\tilde{\mathscr{E}}$. Then $\tilde{\mathscr{F}}$ is a Drinfeld twist corresponding to the same isomorphism, with properties (i)-(iv) of the Drinfeld twist following from the corresponding properties (i)-(iv) of $\mathscr{E}$ and $\tilde{\mathscr{E}}$. By Theorem 4.3.6 there exists a central unitary element $v \in \mathscr{U}(G)$ such that

$$\mathscr{F}\mathscr{E}^*\tilde{\mathscr{E}} = \tilde{\mathscr{F}} = (v \otimes v)\mathscr{F}\hat{\Delta}(v)^*,$$

whence $\tilde{\mathscr{E}} = (v \otimes v)\mathscr{E}\hat{\Delta}(v)^*$.

Furthermore, property (ii) implies $\hat{\varepsilon}(v) = 1$. Property (v) shows that

$$(\hat{S}(v) \otimes \hat{S}(v))\hat{\Delta}(\hat{S}(v))^* = (v \otimes v)\hat{\Delta}(v)^*,$$

that is, $v^*\hat{S}(v)$ is group-like, while property (vi) gives $v\hat{S}(v) = 1$. Therefore the element v^2 is central and group-like, so it is an element of the center of G.

Assume now we have two continuous families $\{\mathscr{E}^q\}_q$ and $\{\tilde{\mathscr{E}}^q\}_q$ of unitaries satisfying properties (i)-(vi) such that $\mathscr{E}^1 = \tilde{\mathscr{E}}^1 = 1$. For every $q > 0$ choose a central unitary element v_q such that $\tilde{\mathscr{E}}^q = (v_q \otimes v_q)\mathscr{E}^q\hat{\Delta}(v_q)^*$. As in the proof of Theorem 4.3.8 we can arrange these unitaries to be continuous in q and assume $v_1 = 1$. Then $\{v_q^2\}_q$ is a continuous family of elements of the finite center of G, hence $v_q^2 = 1$ for all q. But then using continuity once again we get $v_q = 1$. Thus $\tilde{\mathscr{E}}^q = \mathscr{E}^q$. □

It would be interesting to get a more explicit and conceptual description of the family $\{\mathscr{E}^q\}_q$.

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LIST OF SYMBOLS

$A_o(F)$, 5
$A_s(n)$, 6
$A_u(F)$, 5
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The book provides an introduction to the theory of compact quantum groups, emphasizing the role of the categorical point of view in constructing and analyzing concrete examples. The general theory is developed in the first two chapters and is illustrated with a detailed analysis of free orthogonal quantum groups and the Drinfeld-Jimbo q-deformations of compact semisimple Lie groups. The next two chapters are more specialized and concentrate around the Drinfeld-Kohno theorem, presented from the operator algebraic point of view. The book should be accessible to students with a basic knowledge of operator algebras and semisimple Lie groups.

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