Lectures on Elliptic Methods for Hybrid Inverse Problems

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PREFACE

The starting point for these lectures is a course given in Paris between January and March 2014 as part of Chaire Junior of the Fondation Sciences Mathématiques de Paris. This book is designed for a graduate audience, interested in inverse problems and partial differential equations, and we have tried to make it as self-contained as possible.

The analysis of hybrid imaging problems relies on several areas of the theory of PDE together with tools often used to study inverse problems. The full description of the models involved, from the theoretical foundations to the most current developments, would require several volumes and is beyond the scope of these notes, which we designed of a size commensurate with a twenty hour lecture course, the original format of the course. The presentation is limited to simplified settings, so that complete results could be explained entirely. This allows us to provide a proper course, instead of a survey of current research, but it comes at the price that more advanced results are not presented. We have tried to give references to some of the major seminal papers in the area in the hope that the interested reader would then follow these trails to the most current advances by means of usual bibliographical reference libraries.

The physical model most often encountered in this book is the linear Maxwell system of equations. It is of foremost importance in the physics of inverse electromagnetic problems. Compared to the conductivity equation and the Helmholtz equation, the analysis of Maxwell's system is much less developed, and these lectures contain several new results which have been

established while writing this book. In the chapter discussing regularity properties, we focus on the Maxwell system of equations in the time harmonic case. Proofs regarding small volume inhomogeneities are given for Maxwell's system as well.

We introduce the inverse source problem from time-dependent boundary measurements for the wave equation from the classical control theory point of view, leaving aside many deep results related to the geometric control of the wave equation or the Radon transform, or recent developments concerning randomised data. Probabilistic methods are not used, random media are not considered, compressed sensing and other image processing approaches are not mentioned. All these questions would certainly be perfectly natural in this course, but would require a different set of authors. For many of these questions, we refer the reader to the relevant chapters of the Handbook of Mathematical Methods in Imaging [**192**] for detailed introductions and references.

The authors would like to thank the anonymous referee. The manuscript review and the many helpful suggestions it contained have brought us to clarify and improve the presentation of several chapters.

CONTENTS

CHAPTER 1

INTRODUCTION

The inverse problems we discuss are the non-physical counterparts of physics based direct problems. A direct problem is a model of the link from cause to effect, and in this course we shall focus on direct problems modelled by partial differential equations where the effects of a cause are uniquely observable, that is, well posed problems in the sense of Hadamard: from an initial or boundary condition, there exists a unique solution, which depends continuously on the input data [109].

Inverse problems correspond to the opposite problem, namely to find the cause which generated the observed, measured result. Such problems are almost necessarily ill-posed (and therefore non physical). As absolute precision in a measure is impossible, measured data are always (local) averages. A field is measured on a finite number of sensors, and is therefore only known partially. One could say that making a measure which is faithful, that is, which when performed several times will provide the same result, implies filtering small variations, thus applying a compact operator to the full field. Reconstructing the cause from measurements thus corresponds to the inversion of a compact operator, which is necessarily unbounded and thus unstable, except in finite dimension. Schematically, starting from A, a cause (the parameters of a PDE, a source term, an initial condition), which is transformed into B, the solution, by the partial differential equation, and then into C, the measured trace of the

solution, the inversion from C to B is always unstable, whereas the inversion of B to A may be stable or unstable depending on the nature of the PDE, but $B \to A$ is often less severely ill posed than $C \to B$.

As a first fundamental example, let us consider the electrical impedance tomography (EIT) problem, also known as the Calderón problem in the mathematics literature.

1.1. The electrical impedance tomography problem

1.1.1. Measurements on the exterior boundary: the Calderón problem. —
Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz connected bounded domain, where $d \geq 2$ is the dimension of the ambient space.

We consider a real-valued conductivity coefficient $\sigma \in L^\infty(\Omega)$, satisfying

$$(1.1) \qquad \Lambda^{-1} \leq \sigma(x) \leq \Lambda \quad \text{for almost every } x \in \Omega$$

for some constant $\Lambda > 0$.

Definition 1.1. — The *Dirichlet to Neumann map* is

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \quad \langle \Lambda_\sigma \varphi, \psi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x,$$

where $v \in H^1(\Omega)$ is such that $v_{|\partial\Omega} = \psi$ and $u \in H^1(\Omega)$ is the weak solution of

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We need to "prove" this definition, because it apparently depends on the choice of the test function v. Given $v_1, v_2 \in H^1(\Omega)$ with the same trace, namely $v_1 - v_2 \in H_0^1(\Omega)$, from the definition of weak solution we have

$$\int_\Omega \sigma \nabla u \cdot \nabla(v_1 - v_2) \, \mathrm{d}x = 0,$$

thus this definition is proper.

More explicitly, $\Lambda_\sigma \varphi = \sigma \nabla u \cdot \nu_{|\partial\Omega}$, and so Λ_σ maps the applied electric potential φ into the corresponding outgoing current $\sigma \nabla u \cdot \nu_{|\partial\Omega}$. The inverse problem in EIT consists of the reconstruction of σ from voltage-current measurements on $\partial\Omega$. In the case when all possible combinations $(\varphi, \Lambda_\sigma \varphi)$ are available, in the mathematics literature, this inverse problem is called the Calderón problem.

Problem 1.2 (*Calderón problem*). — Determine if the map

$$\sigma \longmapsto \Lambda_\sigma$$

is injective and, in this case, study the inverse $\Lambda_\sigma \mapsto \sigma$.

The injectivity of this map was proved in general in dimension 2 (see [**36**]). In higher dimension, it is established under additional regularity hypotheses on σ [**66**], [**125**], [**202**], with uniqueness for $\sigma \in C^1$ obtained recently in [**110**] and for less regular conductivities in [**111**], [**78**]. Even though the map is injective, the stability of the problem is very poor. It is known to be unstable in general, and furthermore, with the *a priori* additional assumption that $\sigma \in C^m$, $m \geq 2$, the best possible stability estimate is

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leq C \Big(\log \big(1 + \|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_{H^{1/2}, H^{-1/2}}^{-1}\big) \Big)^{-\delta},$$

for some $\delta \in (0, 1)$ depending on d, see [**152**]; thus, only a very coarse reconstruction is possible. To fix ideas, if $C = 1$ and $\delta \approx 1$, a 1 ppm precision leads to an approximation error of 7%, whereas a 1 ppq (one per thousand million of millions) precision leads to an approximation error of 3%.

Remark 1.3. — The general impedance tomography problem considers matrix-valued conductivities, corresponding to anisotropic media. In such generality, Λ is not injective.

1.1.2. The inverse problem with internal data. — The above discussion highlights that absolute impedance measurements, without any prior knowledge of the conductivity, do not seem practical. Since the biological information delivered by the knowledge of the conductivity map is very valuable for a diagnostic point of view (as shown by the large number of publications on this topic in biology and medicine journals), other modalities to measure the conductivity

have been explored. Before we describe some of these hybrid approaches, let us make the following observation.

Proposition 1.4. — *Let $\sigma \in L^{\infty}(\Omega)$ satisfy (1.1) and $u_i \in H^1_{\mathrm{loc}}(\Omega)$ be weak solutions, for $i = 1, \ldots, d$, of*

$$\mathrm{div}(\sigma \nabla u_i) = 0 \quad in \ \Omega.$$

Suppose additionally that in an open subdomain $\Omega' \Subset \Omega$, $\sigma \in H^1(\Omega')$ and for each $u_i \in H^2(\Omega') \cap W^{1,\infty}(\Omega')$, $i = 1, \ldots, d$, and that

$$(1.2) \qquad \det(\nabla u_1, \ldots, \nabla u_d) \geq C \quad a.e. \ in \ \Omega'$$

for some $C > 0$. Then

$$\nabla \log \sigma = -[\nabla u_1, \ldots, \nabla u_d]^{-1} \begin{bmatrix} \mathrm{div}(\nabla u_1) \\ \vdots \\ \mathrm{div}(\nabla u_d) \end{bmatrix} \qquad a.e. \ in \ \Omega'.$$

In particular, σ is known (explicitly) up to a constant multiplicative factor provided that ∇u_i are known in Ω'.

Proof. — Suppose first that $\sigma \in C^{\infty}(\Omega')$; then $u_1, \ldots, u_d \in C^{\infty}(\Omega')$ by elliptic regularity. An explicit calculation immediately yields $\nabla \sigma \cdot \nabla u_i + \sigma \Delta u_i = 0$ in Ω', and in turn

$$\nabla \log \sigma \cdot \nabla u_i = -\Delta u_i \quad in \ \Omega'.$$

In more compact form, by (1.2) this may be rewritten as

$$\nabla \log \sigma = -[\nabla u_1, \ldots, \nabla u_d]^{-1} \begin{bmatrix} \mathrm{div}(\nabla u_1) \\ \vdots \\ \mathrm{div}(\nabla u_d) \end{bmatrix} \quad in \ \Omega'.$$

We conclude by density of smooth functions in Sobolev spaces. $\square$

This result is local, and stability with respect to the derivatives of $u_1, \ldots, u_d$ can be read explicitly. Approximation errors do not spread. Let us now consider the case when only only one datum (instead of d) is available. It is clear that this cannot be enough in general: if σ is a function of $x' = (x_1, \ldots, x_{d-1})$ and $u_d = x_d$, then the above approach gives only

$$\partial_d \log \sigma = 0,$$

which is satisfied by any σ independent of x_d. In general, given any $C^3(\mathbb{R}^d)$ gradient field ∇u such that $m \leq |\nabla u| \leq M$ for some $m, M > 0$, there always exists a C^1 isotropic conductivity δ such that $\mathrm{div}(\delta \nabla u) = 0$, as it is shown in [65]. On the other hand, uniqueness is granted if σ is known on the boundary, at the cost of following gradient flows, provided that a positive Jacobian constraint is satisfied by other, non-measured, gradient fields.

Proposition 1.5. — *Let $\sigma \in C^1(\Omega)$ satisfy (1.1) and $u_i \in C^1(\Omega)$ be weak solutions, for $i = 1, \ldots, d$, of*

$$\mathrm{div}(\sigma \nabla u_i) = 0 \quad in \ \Omega.$$

Suppose that for some nested open subdomains $\Omega' \Subset \widetilde{\Omega} \Subset \Omega$, $(u_1, \ldots, u_d)$ defines a C^1 diffeomorphism from $\widetilde{\Omega}$ to V, so that in particular

$$(1.3) \qquad \det(\nabla u_1, \ldots, \nabla u_d) \geq C \quad in \ \widetilde{\Omega}$$

for some $C > 0$. Suppose that $u_d \in C^2(\widetilde{\Omega})$, and that σ is known on $\partial \Omega'$. Then σ is uniquely determined by ∇u_d in Ω'.

Proof. — For $x_0 \in \partial \Omega'$, consider the following dynamical system (the gradient flow)

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t, x_0) = \nabla u_d \big(X(t, x_0) \big), \quad t \geq 0,$$

$$X(0, x_0) = x_0.$$

As $(u_1, \ldots, u_d)$ is a C^1 diffeomorphism on $\widetilde{\Omega}$, given $y \in \Omega'$ there exists x_0 in $\partial \Omega'$ and $t_0 \in \mathbb{R}$ such that $X(t_0, x_0) = y$ and $X(t, x_0) \in \widetilde{\Omega}$ for all $t \in [0, t_0]$. Now consider

$$f(t) = \log \sigma \big(X(t, x_0) \big).$$

We have, from the same computation as before,

$$f'(t) = \nabla \log \sigma \cdot \nabla u_d = - \mathrm{div} \left(\nabla u_d \big(X(t, x_0) \big) \right),$$

in other words

$$\log \sigma(y) - \log \sigma(x_0) = - \int_0^{t_0} \mathrm{div} \left(\nabla u_d \big(X(t, x_0) \big) \right) \mathrm{d}t. \qquad \square$$

As we see from these two examples, determinant constraints on the gradient fields naturally arise when we wish to use internal gradients to reconstruct the conductivity.

1.2. Some hybrid problem models

The main thread of these lectures is the analysis of the so-called hybrid inverse problems, which are a particular type of inverse problem using internal data coming from the use of coupled-physics phenomena. The gradient fields discussed above are an example of such internal data. In general, the inversion from internal data turns out to be more direct and stable than the corresponding reconstruction from boundary measurements.

The phenomenon used in many of these methods is the dilatation of solids and liquids due to a change of temperature. These phenomena are well known in the physics and mathematics literatures; the appearance of waves in heated fluids or heated elastic bodies was studied by Duhamel (1797–1872). The usefulness of these phenomena for measurement purposes was noted in the mechanics literature in 1962 [**121**]. In the conclusion of that article, they write

> *From the above considerations it appears that as a result of the action of the thermal shock a modified elastic wave and an electromagnetic wave propagate in an elastic medium; there occurs also the radiation of the electromagnetic wave into the vacuum. Besides a manifest theoretical interest in describing the coupled phenomena occurring in an elastic body, the solution obtained is of essential value for the measurement technique.*

The term (and the concept) of *thermoacoustic imaging* appeared in the physics and radiology literature in 1981 [**60**], and *microwave thermoelastic imaging* was introduced a few years later [**143**]. The field then expanded greatly, and seems destined to become a clinically used imaging modality in the not so distant future.

From a practical point of view, these hybrid methods are interesting because they allow measurements that either acoustic imaging or electromagnetic imaging modalities alone would not permit. The electromagnetic radiations (used at low intensity and at not ionising frequency) transmit poorly in biological

tissues, but the heating effect they produce depends directly on the electrical properties of the tissues. The connection between the electrical properties of tissues and their healthiness is well documented; in particular, cancerous tissues are much more conductive, whereas, from an acoustic point of view, they are mostly an aqueous substance, and the contrast with neighbouring healthy tissues is not so important. Thus the stronger pressure waves emitted by the electromagnetic heating allow, if their origin is traced back, to distinguish biologically relevant information with the millimetre precision of the acoustic waves. In other words, the high resolution of acoustic measurements is combined with the high contrast of electromagnetic waves, in order to obtain reconstructions with high resolution as well as high contrast.

Magnetic resonance electrical impedance tomography, originally introduced in the biomedical imaging literature [194] and as a mathematical problem soon after [214], uses magnetic resonance imaging to measure electrical currents generated by the EIT apparatus. Acousto-electric tomography was introduced in [218], reintroduced as ultrasound current source density imaging in [175], and independently described in the mathematical literature under other names during the same period [24], [25]. Here, the ultrasounds are focused to act as the external source of dilatation, whereas the resulting change in the conductivity is measured with usual electrical leads.

The elastic properties of tissues are equally of great practical interest. Not much was available outside of palpation to assess the hardness of tissues until the appearance of hybrid imaging methods, such as sonoelasticity [139] and magnetic resonance elastography [168]. Several other hybrid elastic imaging modalities are currently being developed, see [179] for a review.

Hybrid imaging is not limited to these fields. We refer to [24], [212], [42], [128], [29] for additional methods and further explanations on some of the models (briefly) described below.

In most hybrid imaging modalities, the reconstruction is performed in two steps. First, internal measurements are recovered inside the domain of interest. These data usually take the form of a functional depending on the unknowns of the problem in a very nonlinear way, also through the solutions of the PDE modelling the direct problem. In a second step, the unknown coefficients have

to be reconstructed. We present below a few examples of hybrid inverse problems, which will be studied in detail in the second part of the book. In order to study these imaging methods rigorously, a number of mathematical questions must be answered. We will highlight some of them, which will be the focus of the first part of these lectures.

1.2.1. Magnetic resonance electric impedance tomography – current density impedance imaging. — In these modalities, the magnetic field generated by artificially induced electric currents is measured with a magnetic resonance imaging (MRI) scanner. Either one or all components of the magnetic field H are measured. We speak of *magnetic resonance electric impedance tomography* (MREIT) in the first case and of *current density impedance imaging* (CDII) in the latter. Here we consider only CDII.

In the setting of the linear Maxwell system of equations

$$\begin{cases} \operatorname{curl} E = \mathrm{i}\omega H & \text{in } \Omega, \\ \operatorname{curl} H = -\mathrm{i}(\omega\varepsilon + \mathrm{i}\sigma)E & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases}$$

in a first step the internal magnetic field H generated by the boundary value φ is measured with an MRI scanner. In a second step, the electric permittivity ε and the conductivity σ have to be reconstructed from the knowledge of several measurements of H^i corresponding to multiple boundary illuminations φ_i. We shall see that this step requires linearly independent electric fields: this condition corresponds to the Jacobian constraint for the electric potentials.

We also study the scalar approximation in the limit $\omega \to 0$, namely the conductivity equation. More precisely, taking $\omega = 0$ in the above system allows us to write $E = \nabla u$ for some electric potential u, since by the first equation E is irrotational (provided that Ω is simply connected). Thus, the second equation yields

$$-\operatorname{div}(\sigma\nabla u) = 0 \quad \text{in } \Omega.$$

Moreover, the second equation allows us to measure the internal current density via

$$J = \sigma\nabla u = \operatorname{curl} H \quad \text{in } \Omega$$

from the knowledge of the internal magnetic field. If multiple measurements are performed, then we measure $J_i = \sigma \nabla u^i$ for several applied boundary voltages. In a second step, the unknown conductivity has to be reconstructed from the knowledge of the currents J_i. Except for the factor σ, this problem is very similar to the one considered in §1.1.2, where the internal data simply consisted of the gradient fields ∇u^i. It is therefore expected that the Jacobian constraint (1.2) will play an important role in the inversion. (In fact, in the three dimensional case, only two linearly independent gradients will be needed.)

To summarise, this hybrid problem consists of the following two steps:

- ▷ The reconstruction of the magnetic field H (and hence of J), from MRI data.
- ▷ The reconstruction of σ from the knowledge of the internal current densities J_i (or directly from H^i in the case of Maxwell's system).

1.2.2. Acousto-electric tomography. — The main feature of coupled-physics, or hybrid, inverse problems is the use of two types of waves simultaneously. Acousto-electric tomography (AET) belongs to a class of hybrid problems in which the first type of wave is used to perturb the medium while the second wave is used to make measurements. In AET, ultrasound waves are used to perturb the domain, while electrical measurements are taken via the standard EIT setup discussed in the previous section. Physically, the pressure change caused by the ultrasounds will modify the density of the tissue, which in turn affects the electric conductivity. The availability of the electrical measurements in both the unperturbed and perturbed case allows one to obtain internal data, as we now briefly discuss.

Ultrasounds may be used in different ways to perturb the domain. Depending on the particular experimental configuration, different reconstruction methods need to be used in order to obtain the internal data. However, at least in theory, these data are independent of the particular setting: they consist in the pointwise electrical energy

$$(1.4) \qquad H(x) = \sigma(x) \left| \nabla u(x) \right|^2, \quad x \in \Omega,$$

where σ is the conductivity and u the electric potential.

We consider only the case of focused ultrasonic waves. Focusing an ultrasonic wave on a small domain B_x centred around a point x will change the conductivity in B_x, in a quantifiable way. For a fixed applied boundary potential, the corresponding current can be measured on $\partial\Omega$. These measurements are performed in the unperturbed situation, namely when the ultrasound waves are not used, and in the perturbed case. When we compute the cross-correlation of these measurements on $\partial\Omega$, we expect it to reflect local information of the conductivity near x.

The precise connection between cross-correlation and local quantities needs to be clarified. With an integration by parts it is possible to express the cross-correlation of the boundary measurements with a local expression of $u_x - u$ near x, where u_x is the electrical potential created in the perturbed case. Assuming that the size of the perturbation is small, we may write an asymptotic expansion of $u_x - u$ near x. At first order, such an expansion yields the internal data given in (1.4). More generally, using multiple measurements, it is possible to recover

$$H_{ij}(x) = \sigma(x)\nabla u_i(x) \cdot \nabla u_j(x), \quad x \in \Omega.$$

In the quantitative step of AET, the unknown conductivity σ has to be reconstructed from these measurements. Note that

$$H_{ij} = S_i \cdot S_j,$$

where $S_i = \sqrt{\sigma}\nabla u_i$ is nothing other than the interior current density J_i considered in CDII, up to a factor $\sqrt{\sigma}$. We shall show that if the Jacobian constraint (1.2) is satisfied, then it is possible to recover S_i, $i = 1, \ldots, d$, from the knowledge of their pairwise scalar products. Then, the conductivity can be recovered with a method similar to the one used in CDII.

To sum up, the two step inverse problem in AET consists of

▷ extracting localised information about the unperturbed gradient field ∇u from the knowledge of u_x and u on the boundary, by using a local asymptotic expansion;

▷ and in reconstructing the conductivity σ from these internal data.

1.2.3. Thermoacoustic tomography. — Thermoacoustic tomography (TAT) is a hybrid imaging modality where electromagnetic waves are combined with ultrasounds [**208**]. As discussed above, part of the electromagnetic radiation is absorbed by the tissues, and hence transformed into heat. The increase in temperature causes an expansion of the medium, which in turn creates acoustic waves. In TAT, waves in the microwave range are usually used to illuminate the medium.

If we consider the problem in a bounded domain Ω with Dirichlet boundary conditions, the acoustic pressure p satisfies

$$\begin{cases} c(x)^2 \Delta p - \partial_{tt}^2 p = 0 & \text{in } \Omega \times (0, T), \\ p(x, 0) = H(x) & \text{in } \Omega, \\ \partial_t p(x, 0) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega \times (0, T), \end{cases}$$

where c is the sound speed of the medium and H is the absorbed electromagnetic energy. The available measured data is the quantity

$$\partial_\nu p(x, t), \quad x \in \Gamma, \ t \in [0, T],$$

which is obtained via acoustic sensors positioned on a part of the boundary $\Gamma \subseteq \partial\Omega$. In a first step, from these measurements the initial source H has to be reconstructed. This is the typical observability/control problem for the wave equation: we wish to recover the initial condition from boundary measurements of the solution over time. The reconstructed internal data take the form

$$H(x) = \sigma(x) |u(x)|^2, \quad x \in \Omega,$$

where σ is the spatially varying conductivity of the medium and u is the (scalar) electric field and satisfies the Helmholtz equation

$$\begin{cases} \Delta u + (\omega^2 + i\omega\sigma)u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

In this context, this PDE should be seen as a scalar approximation of the full Maxwell system.

In a second step, the unknown conductivity σ has to be reconstructed from the knowledge of H. Multiple measurements, corresponding to several boundary illuminations φ_i, can be taken. When compared to the previous hybrid problems considered, the internal energy $H = \sigma|u|^2$ has a different structure, as it does not involve gradient fields. However, similar ideas to those used before can be applied and σ can be uniquely and stably recovered provided that a generalised Jacobian constraint is verified.

Without going further in this discussion, we see that this hybrid problem contains two consecutive inverse problems:

▷ A hyperbolic source reconstruction problem for the wave equation, to derive $H = \sigma|u|^2$ from the measured pressure data.
▷ An elliptic problem with internal data, to recover σ from the knowledge of the electromagnetic power densities $H_i = \sigma|u_i|^2$.

1.2.4. Dynamic elastography. — In contrast to the previous model, shear wave elastography (or acoustic radiation force impulse, or supersonic shear imaging) usually uses sources that induce shear waves. Such waves travel slowly, and therefore on the time-scale of the shear waves, the terms that would be captured by a displacement which is the gradient of a potential are negligible (in a time averaged sense, or equivalently in filtered Fourier sense) as the ratio of the propagation speed is $\sqrt{\lambda/\mu} \approx 22$, where λ and μ are the Lamé parameters, μ being the shear modulus. Assuming that the source is generated by a single frequency mechanical wave, the model is then (after a Fourier transform in time)

$$\operatorname{div}(\mu\nabla u_s) + \rho\,\omega^2 u_s = 0,$$

(see *e.g.* [154], [153], [203], [102]). The shear wave displacement can be recovered by different methods. If a magnetic resonance imager is used, u_s is delivered (almost) directly by a careful synchronisation of the frequency of the imaging magnetic field and currents with that of the shear wave [168]. In other modalities, the shear wave behaves as a stationary source for the acoustic waves;

reconstructing the source of these acoustic waves then leads to the reconstruction of the variations of u_s from external measurements (the so called ultrasound Doppler effect) [107], [179]. The two embedded problems are in this case:

▷ A hyperbolic source reconstruction problem for a wave equation, to derive u_s from the measured acoustic data (this step being avoided in the case of MRE measurements).

▷ An elliptic inverse problem with internal data, to recover μ and ρ from the knowledge of the displacement u_s.

1.2.5. Photoacoustic tomography. — Photoacoustic tomography (PAT) is a particular instance of thermoacoustic tomography where high frequency electromagnetic waves (lasers) are used instead of low frequency ones (microwaves) [208]. Thus, the physical coupling and the model coincide with the ones discussed above for TAT. The only difference is in the form of the internal data, which in PAT are

$$H(x) = \Gamma(x)\mu(x)u(x), \quad x \in \Omega,$$

where Γ is the Grüneisen parameter, μ is the light absorption and u is the light intensity.

In a first step, H has to be recovered from the acoustic measurements on part of $\partial\Omega$: this can be achieved exactly as in TAT. In a second step, we need to reconstruct the light absorption μ from the knowledge of several internal data $H_i = \Gamma\mu u_i$. In the diffusion approximation for light propagation, u satisfies the second-order elliptic PDE

$$-\operatorname{div}(D\nabla u) + \mu u = 0 \quad \text{in} \ \ \Omega,$$

and the reconstruction becomes an elliptic inverse problem with internal data. Thus, the inversion will be similar to those related to the modalities discussed before.

1.3. Selected mathematical problems arising from these models

The five examples surveyed above are different in terms of the physical phenomena involved, both with respect to the output measured quantities and with respect to the input generating sources. At the level of mathematical modelling, they share several similarities.

The *observability of the wave equation* often arises. In thermoacoustic tomography, ultrasound elastography and photoacoustic tomography, the first step corresponds to the reconstruction of the initial condition of a wave equation in a bounded domain from the knowledge of its solution measured on the boundary over time.

The physical quantities involved in the above examples are typically understood to be defined pointwise, and the formal computations performed become meaningful thanks to *regularity estimates*. The models involved for the "second step" are either quasistatic or time harmonic problems, for which elliptic regularity theory can be applied.

This leads one to wonder how general such developments are, and to investigate what happens when the apposite assumptions are violated by a perturbation. *Small volume inclusions* are an example of such perturbation. In the context of elliptic boundary value problems, a small inclusion actually appears in the derivation of the acousto-electric model. In the context of the wave equation, controlling the influence of a defect is related to deriving a *scattering estimate*.

Another common feature is the appearance of *positivity constraints*. For all these methods to provide meaningful data, the internal measurements obtained during the first step must be non zero. The key issue is that these data, whether it is an internal heating by the Joule effect or compression waves or electrical currents, are only indirectly controlled by the practitioner, who imposes a boundary condition (or an incident field) outside the medium. The question is therefore whether one can indeed guarantee certain non-vanishing conditions (*e.g.* a Jacobian as in (1.2)) independently of the unknown parameters by an appropriate choice of the boundary conditions.

1.4. Outline of the following chapters

Let us now briefly discuss the content of this book. In Part I, we focus on the rigorous exposition of some mathematical tools which prove useful to address the mathematical challenges mentioned in the previous section. In Part II, we apply these methods to various hybrid imaging modalities.

The focus of Chapter 2 is the observability of the wave equation. We prove the observability inequality under certain sufficient conditions on the domain and on the sound speed due to Lions, and discuss the link with the Hilbert uniqueness method. All the material presented here is classical, but the exposition follows the point of view of inverse problems, from the uniqueness of the reconstruction of the initial condition to the possible practical implementation of the inversion.

We shall use either Maxwell's system or some scalar approximation of this system as a model for the underlying physics. In Chapter 3, we study the regularity theory for the linear time-harmonic Maxwell system. We prove both $W^{1,p}$ and $C^{0,\alpha}$ estimates for the electromagnetic fields under natural, and sometimes minimal, assumptions on the coefficients. Only interior regularity estimates are derived for simplicity. Most of the material presented in this chapter is new, and relies on the application of standard elliptic regularity theory results to the vector and scalar potentials of the electromagnetic fields obtained by the Helmholtz decomposition.

These regularity results find applications also in the study of small volume perturbations for Maxwell's system, which is carried out in Chapter 4 (as a corollary, we derive the well-known result for the conductivity equation, needed in acousto-electric tomography). The strategy used for the regularity of the electromagnetic fields carries over to this case: it turns out that the problem can be simplified to considering coupled elliptic equations for the potentials. We can then apply methods developed for the conductivity problem. The results presented here have been known for a few years, in a less general setting. The approach presented here shortens the proof significantly.

In Chapter 5, we present some results on scattering estimates for the Helmholtz equation in two and three dimensions. We consider the particular case of a single ball scatterer in a homogeneous medium, and derive estimates

for the near and far scattered field. The dependence of these estimates on the radius and on the contrast of the inclusion and on the operating frequency of the incident field will be explicit. These results were not stated in this form previously, but the ingredients of their proofs were already known. The proofs presented here simplify the original arguments.

The last three chapters of Part I present four different techniques for the boundary control of elliptic PDE in order to enforce certain non-zero constraints for the solutions, such as a non-vanishing Jacobian, as discussed above.

The focus of Chapter 6 is the Jacobian constraint for the conductivity equation. We first review some extensions of the Radó-Kneser-Choquet theorem for the conductivity equation in two dimensions, and give a self-contained proof of the result. A quantitative version of this result is derived by a compactness argument. Next, we show that the result is not true in dimensions higher than two by means of a new explicit counter-example. More precisely, it is proven that for any boundary value there exists a conductivity such that the Jacobian of the corresponding solutions changes its sign in the domain. This constructive result was recently obtained in dimension three, and it is extended here to the higher dimensional case. A new corollary of this result for finite families of boundary conditions is also provided.

In Chapter 7, we discuss two other techniques for the construction of boundary conditions so that the corresponding solutions to certain elliptic PDE satisfy some predetermined, non-vanishing constraints inside the domain: the complex geometric optics (CGO) solutions and the Runge approximation property. Except for the proof of the existence and regularity of CGO solutions and the unique continuation property for elliptic PDEs, the exposition is self-contained and reviews known results related to these topics. Applications to the constraints arising from hybrid imaging are discussed.

Another method for the construction of suitable boundary values is discussed in Chapter 8, and is based on the use of multiple frequencies. As such, this technique is applicable only with frequency-dependent, or time harmonic, PDE. The advantage over the methods discussed in the previous chapter lies in the explicit construction of the boundary values, often independently of the unknown parameters. A self-contained exposition of this

approach is presented in this chapter. For simplicity, we restrict ourselves to the simpler case of the Helmholtz equation with complex potential.

Chapter 9 and Chapter 10 are grouped into Part II of these lectures, where the various results and methods introduced in Part I are put to use.

Chapter 9 is the first chapter of Part II and deals with the first step of the hybrid inverse problems introduced in Section 1.2. The physical aspects of these modalities are only mentioned, and not analysed in detail. The focus of the chapter is on the application of the mathematical tools introduced before, in particular of the observability of the wave equation discussed in Chapter 2 and of the small inclusion expansions (Chapter 4), in order to obtain the internal data.

The reconstruction of the unknown parameters from the internal data for these hybrid modalities is discussed in Chapter 10 . The focus is on explicit inversion methods, based on the non-vanishing constraints for PDEs, which were presented in Chapters 6, 7, 8. The issue of stability is considered precisely in one case, and only mentioned for the other modalities. Since these reconstruction algorithms always require differentiation of the data, carefully designed regularisation or optimisation schemes are needed for their numerical implementation. This fundamental aspect is not considered here, and the reader is referred to the extensive literature on the topic.

We depart from the theorem / proof formalism in Part II. The derivations of the relevant physical quantities in Chapter 9 are in some cases reasoned rather than proved. While the content of Chapter 10 could be written in the format used in the first part of the book, we felt it could distract the reader from the purpose of this last part, which is to explain reconstruction methods in a straightforward manner, and highlight how various tools developed in Part I are used.

PART I

MATHEMATICAL TOOLS

<div style="text-align: center">

CHAPTER 2

THE OBSERVABILITY OF THE WAVE EQUATION

</div>

2.1. Introduction

This chapter briefly discusses what was described in the previous chapter as a hyperbolic source reconstruction problem for the wave equation. Namely, we focus on the following PDE,

$$\begin{cases} c^{-2}\partial_{tt}p - \Delta p = 0 & \text{in } (0,T) \times \Omega, \\ p(0,x) = A(x) & \text{in } \Omega, \\ \partial_t p(0,x) = 0 & \text{in } \Omega, \end{cases}$$

which, as we saw, arises in thermoacoustic, photoacoustic and ultrasound elastography models. The inverse problem at hand is: given a measure over a certain duration of a pressure related quantity on $\partial\Omega$, or on a part of $\partial\Omega$, recover the initial pressure $p(x,0)$, that is, $A(x)$. The function $c(x)$ represents the sound velocity, which may vary spatially.

Note that, in this form, this problem is not well posed: the boundary condition is missing. If one assumes that the pressure wave propagates freely outside of Ω into the whole space, this problem is profoundly connected with the generalised Radon transform. We refer the reader to [127] and [199] to explore this question. We will present another point of view, discussed in [26], [130], [116], and consider the problem in a confined domain, and assume that on the boundary (of the soft tissues), the pressure is either reflected (Neumann)

or absorbed (Dirichlet). In this last case, the boundary condition is

$$p = 0 \quad \text{on } (0,T) \times \partial\Omega,$$

and what is measured is the outgoing flux $\partial_\nu p$ on a part of the boundary $\Gamma \subseteq \partial\Omega$. More precisely, this inverse problem may be formulated as follows.

Problem 2.1. — Let p be the weak solution of

$$(2.1) \qquad \begin{cases} c^{-2}\partial_{tt}p - \Delta p = 0 & \text{in } (0,T) \times \Omega, \\ p(0, \bullet) = A & \text{in } \Omega, \\ \partial_t p(0, \bullet) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

where c is a positive function defined on Ω. Supposing that the trace of $\partial_\nu p$ is measured on an open subset Γ of $\partial\Omega$ for all $t \in (0,T)$, find the initial condition A in Ω.

The sound velocity could itself be considered an unknown of the problem: we will not discuss this aspect here. The observability and the boundary control of the wave equation is a sub-subject of the analysis of PDE in itself; we will not attempt to provide a full review of the many advances on this question. We refer to the celebrated classic texts [**189**], [**144**], [**145**], [**126**], [**133**] for a general presentation, and to the survey paper [**94**] for more recent advances and references to (some of) the authoritative authors in this field. The purpose of this chapter is to describe briefly some of the classical results regarding this problem.

2.2. Well-posedness and observability

In this chapter, $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with C^2 boundary $\partial\Omega$ and $d \geq 2$. Unless otherwise stated, the function spaces used in this chapter consist of real-valued functions. The model problem we consider is

$$(2.2) \qquad \begin{cases} c^{-2}\partial_{tt}\varphi - \Delta\varphi = 0 & \text{in } (0,T) \times \Omega, \\ \varphi(0, \bullet) = \varphi_0 & \text{in } \Omega, \\ \partial_t\varphi(0, \bullet) = \varphi_1 & \text{in } \Omega, \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

which corresponds to Problem 2.1 when $\varphi_0 = A$ and $\varphi_1 = 0$.

Let us first recall the appropriate functional analysis context for this problem. The following result is classical. Under slightly stronger assumptions, it can be found in [**95**, Chapter 7.2].

Proposition 2.2. — *Let $c \in W^{1,\infty}(\Omega; \mathbb{R}_+)$ be such that $\log c \in W^{1,\infty}(\Omega)$ and $T > 0$.*

▷ *For every $\varphi_0 \in H_0^1(\Omega)$ and $\varphi_1 \in L^2(\Omega)$ there exists a unique weak solution $\varphi \in L^2(0, T; H_0^1(\Omega))$, with $\partial_t \varphi \in L^2(0, T; L^2(\Omega))$, of (2.2). Furthermore, we have $\varphi \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.*

▷ *If we define the energy of the system by*

$$E(t) = \tfrac{1}{2} \int_\Omega c^{-2} \big(\partial_t \varphi(t, x) \big)^2 + \big| \nabla \varphi(t, x) \big|^2 \, \mathrm{d}x, \quad t \in (0, T),$$

then $E(t) = \tfrac{1}{2} \int_\Omega c^{-2} \varphi_1^2 + |\nabla \varphi_0|^2 \, \mathrm{d}x, \quad t \in (0, T)$.

Remark 2.3. — In view of this result, problem (2.1) is well posed for any $T > 0$, $c \in W^{1,\infty}(\Omega)$ with $\min_{\overline{\Omega}} c > 0$, and $A \in H_0^1(\Omega)$.

Recall that our measured data is a normal derivative on the boundary. The trace of the derivative of a $H_0^1(\Omega)$ function is *a priori* defined only in $H^{-1/2}(\partial\Omega)$. However, in our case it turns out that it is in $L^2(\partial\Omega)$. This result is associated to several names in the literature, namely Rellich, Pohozaev, and Morawetz, and is detailed in the following lemma.

Lemma 2.4. — *Let $c \in W^{1,\infty}(\Omega)$ be such that $\min_{\overline{\Omega}} c > 0$, $T > 0$ and Γ be an open subset of $\partial\Omega$. The map*

$$(2.3) \qquad D_\Gamma : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow L^2\big((0, T) \times \Gamma\big), \quad (\varphi_0, \varphi_1) \longmapsto \partial_\nu \varphi,$$

where φ is the solution of (2.2), is continuous.

A proof of this lemma is given at the end of the chapter for the reader's convenience.

Corollary 2.5. — *Under the hypotheses of Lemma 2.4, the map*

$$d_\Gamma : H_0^1(\Omega) \longrightarrow L^2((0, T) \times \Gamma), \quad A \longmapsto \partial_\nu p,$$

where p is the solution of (2.1), is continuous.

This result completes the study of the direct problem associated to (2.1). The corresponding inverse problem, namely Problem 2.1, consists of the reconstruction of A from $\partial_\nu p$. In order to achieve this we need more, namely that d_Γ is in fact injective with bounded inverse for T large enough. More generally, we consider the invertibility of the map D_Γ.

Definition 2.6. *— The initial value problem* (2.2) *is* observable *at time T from Γ if there exists a constant $C > 0$ such that for any $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution φ of* (2.2) *satisfies the observability inequality*

$$(2.4) \qquad \|\varphi_1\|_{L^2(\Omega)} + \|\varphi_0\|_{H_0^1(\Omega)} \leq C \|\partial_\nu \varphi\|_{L^2((0,T)\times\Gamma)}.$$

Remark 2.7. *— If the initial value problem* (2.2) *is observable at time T from Γ, then the map d_Γ is invertible with bounded inverse: A is uniquely and stably determined by the boundary data $\partial_\nu p$ on $(0,T) \times \Gamma$. This solves Problem 2.1.*

It is natural to ask for which Ω, Γ, c and T condition (2.4) holds. This question was answered completely in generic smooth domains in [**53**], [**68**], where it is shown that exact controllability and geometric controllability (see [**155**], [**156**]) are equivalent. An informal definition of geometric controllability is that every ray of geometric optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ should meet Γ in time less than T at a non diffractive point. The exact definition of geometric control on generic domains is beyond the scope of these lectures. We present a stronger sufficient condition, due to Lions [**145**], [**146**].

Theorem 2.8. *— Under the hypotheses of Lemma 2.4, if there exists $x_0 \in \mathbb{R}^d$ such that*

$$(2.5) \qquad \left\{ x \in \partial\Omega : (x - x_0) \cdot \nu > 0 \right\} \subseteq \Gamma$$

and $1 > 2 \|(\nabla(\log c) \cdot (x - x_0))_+\|_{L^\infty(\Omega)}$, then, for all times

$$T > T_0 = \frac{2 \sup_{x\in\Omega} |x - x_0| \cdot \|c^{-1}\|_{L^\infty(\Omega)}}{(1 - 2\|(\nabla(\log c) \cdot (x - x_0))_+\|_{L^\infty(\Omega)})},$$

the system (2.2) *is observable at time T from Γ.*
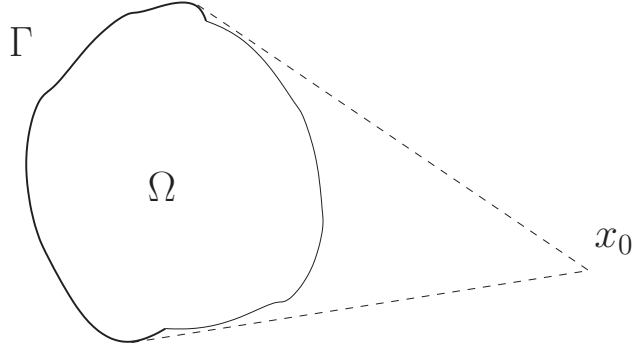
FIGURE 2.1. A sufficient boundary portion Γ when $\nabla c = 0$.

Proof. — Let φ be the solution of (2.2) with initial conditions φ_0 and φ_1. By Proposition 2.2, for every $t \in [0, T]$ we have

$$(2.6) \qquad \frac{1}{2} \int_\Omega \left(c^{-2} (\partial_t \varphi)^2 + |\nabla \varphi|^2 \right) dx = E_0 = \frac{1}{2} \int_\Omega \left(c^{-2} \varphi_1^2 + |\nabla \varphi_0|^2 \right) dx.$$

Testing (2.2) against φ, we obtain

$$(2.7) \qquad \int_\Omega c^{-2} \partial_t \varphi \varphi \Big|_0^T dx = \int_{(0,T) \times \Omega} \left(c^{-2} (\partial_t \varphi)^2 - |\nabla \varphi|^2 \right) dt \, dx.$$

The method of proof is similar to the one we use later for the proof of Lemma 2.4. We test $c^{-2} \partial_{tt} \varphi$ against $(x - x_0) \cdot \nabla \varphi$ and obtain

$$(2.8) \qquad \int_{(0,T) \times \Omega} c^{-2} \partial_{tt} \varphi \, (x - x_0) \cdot \nabla \varphi \, dt \, dx = \int_\Omega \partial_t \varphi \left(c^{-2} (x - x_0) \cdot \nabla \varphi \right) \Big|_0^T dx$$

$$- \frac{1}{2} \int_{(0,T) \times \Omega} c^{-2} (x - x_0) \cdot \nabla \left((\partial_t \varphi)^2 \right) dt \, dx$$

$$= \int_\Omega \partial_t \varphi \left(c^{-2} (x - x_0) \cdot \nabla \varphi \right) \Big|_0^T dx$$

$$- \frac{1}{2} \int_{(0,T) \times \Omega} (x - x_0) \cdot \nabla \left(c^{-2} (\partial_t \varphi)^2 \right) dt \, dx$$

$$+ \frac{1}{2} \int_{(0,T) \times \Omega} \nabla c^{-2} \cdot (x - x_0) (\partial_t \varphi)^2 \, dt \, dx.$$

Note that $\partial_t \varphi = 0$ on $(0,T) \times \partial\Omega$ as $\varphi = 0$ on $(0,T) \times \partial\Omega$. Performing an integration by parts and then using (2.7), we have

$$(2.9) \quad -\frac{1}{2} \int_{(0,T)\times\Omega} (x-x_0) \cdot \nabla\left(c^{-2}(\partial_t \varphi)^2\right) dt\,dx$$

$$= \frac{1}{2}d \int_{(0,T)\times\Omega} c^{-2}(\partial_t \varphi)^2 \, dt\,dx$$

$$= \frac{1}{2} \int_{(0,T)\times\Omega} c^{-2}(\partial_t \varphi)^2 \, dt\,dx + \frac{1}{2}(d-1) \int_{(0,T)\times\Omega} |\nabla\varphi|^2 \, dt\,dx$$

$$+ \frac{1}{2}(d-1) \int_{\Omega} c^{-2}\partial_t \varphi\varphi \Big|_0^T \, dx.$$

Testing $-\Delta\varphi$ against $(x-x_0) \cdot \nabla\varphi$, we have

$$-\int_{(0,T)\times\Omega} \Delta\varphi\,(x-x_0) \cdot \nabla\varphi\,dt\,dx = -\int_{(0,T)\times\partial\Omega} (\partial_\nu \varphi)^2 \left[(x-x_0) \cdot \nu\right] dt\,d\sigma$$

$$+ \int_{(0,T)\times\Omega} \nabla\varphi \cdot \nabla\left((x-x_0) \cdot \nabla\varphi\right) dt\,dx$$

$$= -\int_{(0,T)\times\partial\Omega} (\partial_\nu \varphi)^2 \left[(x-x_0) \cdot \nu\right] dt\,d\sigma$$

$$+ \int_{(0,T)\times\Omega} |\nabla\varphi|^2 \, dt\,dx + \frac{1}{2} \int_{(0,T)\times\Omega} (x-x_0) \cdot \nabla\left(|\nabla\varphi|^2\right) dt\,dx,$$

which yields in turn

$$-\int_{(0,T)\times\Omega} \Delta\varphi\,(x-x_0) \cdot \nabla\varphi\,dt\,dx = -\frac{1}{2} \int_{(0,T)\times\partial\Omega} (\partial_\nu \varphi)^2 \left[(x-x_0) \cdot \nu\right] dt\,d\sigma$$

$$-\frac{1}{2}(d-2) \int_{(0,T)\times\Omega} |\nabla\varphi|^2 \, dt\,dx.$$

Therefore, by (2.8) and (2.9), since $-\Delta\varphi + c^{-2}\partial_{tt}\varphi = 0$ in Ω, we obtain

$$(2.10) \quad \frac{1}{2}\int_{(0,T)\times\partial\Omega}(\partial_\nu\varphi)^2\big[(x-x_0)\cdot\nu\big]\,dt\,d\sigma$$

$$= \frac{1}{2}\int_{(0,T)\times\Omega}\big(c^{-2}(\partial_t\varphi)^2 + |\nabla\varphi|^2\big)\,dt\,dx$$

$$-\int_{(0,T)\times\Omega}(\nabla\log c)\cdot(x-x_0)c^{-2}(\partial_t\varphi)^2\,dt\,dx$$

$$+\int_\Omega c^{-2}\partial_t\varphi\big((x-x_0)\cdot\nabla\varphi + \frac{1}{2}(d-1)\varphi\big)\,\Big|_0^T\,dx.$$

Thanks to the conservation of energy (2.6), we can bound the first two terms from below, namely

$$(2.11) \quad \frac{1}{2}\int_{(0,T)\times\Omega}(c^{-2}(\partial_t\varphi)^2 + |\nabla\varphi|^2$$

$$- 2\big((\nabla\log c)\cdot(x-x_0)\big)c^{-2}(\partial_t\varphi)^2\big)\,dt\,dx$$

$$\geq TE_0\Big(1 - 2\big\|\big((\nabla\log c)\cdot(x-x_0)\big)_+\big\|_{L^\infty(\Omega)}\Big).$$

Let us focus on the last term in (2.10). For every $\lambda > 0$ we have

$$\Big|\int_\Omega c^{-2}\partial_t\varphi\big((x-x_0)\cdot\nabla\varphi + \frac{1}{2}(d-1)\varphi\big)\,dx\Big|$$

$$\leq \|c^{-1}\|_{L^\infty(\Omega)}\Big(\frac{\lambda}{2}\int_\Omega c^{-2}(\partial_t\varphi)^2\,dx$$

$$+ \frac{1}{2\lambda}\int_\Omega\big|(x-x_0)\cdot\nabla\varphi + \frac{1}{2}(d-1)\varphi\big|^2\,dx\Big).$$

By expanding the second square and integrating by parts we obtain

$$\frac{1}{2}\int_{\Omega}\left|(x-x_0)\cdot\nabla\varphi+\tfrac{1}{2}(d-1)\varphi\right|^2\mathrm{d}x$$

$$=\frac{1}{2}\int_{\Omega}\left|(x-x_0)\cdot\nabla\varphi\right|^2\mathrm{d}x+\frac{1}{2}\int_{\Omega}\tfrac{1}{2}(d-1)(x-x_0)\cdot\nabla(\varphi^2)\,\mathrm{d}x$$

$$+\frac{1}{2}\int_{\Omega}\tfrac{1}{4}(d-1)^2\varphi^2\,\mathrm{d}x$$

$$=\frac{1}{2}\int_{\Omega}\left|(x-x_0)\cdot\nabla\varphi\right|^2\mathrm{d}x-\tfrac{1}{8}(d^2-1)\int_{\Omega}\varphi^2\,\mathrm{d}x$$

$$\le\sup_{x\in\Omega}|x-x_0|^2\left(E_0-\frac{1}{2}\int_{\Omega}c^{-2}(\partial_t\varphi)^2\,\mathrm{d}x\right).$$

We balance both terms with $\lambda=\sup_{x\in\Omega}|x-x_0|$ and obtain

$$(2.12)\qquad\int_{\Omega}c^{-2}\,\partial_t\,\varphi\Big((x-x_0)\cdot\nabla\varphi+\tfrac{1}{2}(d-1)\varphi\Big)\,\Big|_0^T\mathrm{d}x$$

$$\ge-2\|c^{-1}\|_{L^\infty(\Omega)}E_0\sup_{x\in\Omega}|x-x_0|,$$

$$=-\Big(1-2\big\|(\nabla\log c\cdot(x-x_0))_+\big\|_{L^\infty(\Omega)}\Big)T_0\,E_0$$

and combining (2.10), (2.11) and (2.12) we obtain

$$\frac{1}{2}\int_{(0,T)\times\partial\Omega}(\partial_\nu\varphi)^2\big[(x-x_0)\cdot\nu\big]\,\mathrm{d}t\,\mathrm{d}\sigma$$

$$\ge\Big(1-2\|(\nabla\log c\cdot(x-x_0))_+\|_{L^\infty(\Omega)}\Big)(T-T_0)E_0,$$

which gives our result. Indeed, by (2.5) we have

$$\int_{(0,T)\times\partial\Omega}(\partial_\nu\varphi)^2\big[(x-x_0)\cdot\nu\big]\,\mathrm{d}t\,\mathrm{d}\sigma\le\int_{(0,T)\times\Gamma}(\partial_\nu\varphi)^2\big[(x-x_0)\cdot\nu\big]\,\mathrm{d}t\,\mathrm{d}\sigma$$

$$\le\sup_{x\in\Omega}|x-x_0|\int_{(0,T)\times\Gamma}(\partial_\nu\varphi)^2\,\mathrm{d}t\,\mathrm{d}\sigma,$$

and

$$E_0=\frac{1}{2}\int_{\Omega}\big(c^{-2}\varphi_1^2+|\nabla\varphi_0|^2\big)\,\mathrm{d}x\ge C'\big(\|\varphi_1\|_{L^2(\Omega)}^2+\|\varphi_0\|_{H_0^1(\Omega)}^2\big)$$

for some $C' > 0$ depending only on $\|c\|_{L^\infty(\Omega)}$ and Ω. As a consequence, we have

$$\|\partial_\nu \varphi\|_{L^2((0,T)\times\Gamma)} \geq C\left(\|\varphi_1\|_{L^2(\Omega)} + \|\varphi_0\|_{H_0^1(\Omega)}\right)$$

with $C = C''\sqrt{T/T_0 - 1}$ for some $C'' > 0$ depending only on $\|c\|_{L^\infty(\Omega)}$, $\|c^{-1}\|_{L^\infty(\Omega)}$ and Ω. This concludes the proof. $\qquad\square$

Remark 2.9. — If we apply Lions' Γ condition (2.5) in a ball of radius R with $c \equiv 1$, and choose x_0 to be its centre, we find the minimal time to control from the full boundary $T_0 = 2R$, which is in agreement with the Geometric Control Condition. In general, for a constant velocity, we see that a sufficient portion of a ball is more than half of its boundary (pushing x_0 towards infinity); this also agrees with the sharp condition as it captures all radially bouncing rays.

Remark 2.10. — The second part of the sufficient condition, namely

$$1 > 2\left\|\left(\nabla \log c \cdot (x - x_0)\right)_+\right\|_{L^\infty(\Omega)},$$

is relevant only for variable velocities c. It is not optimal, but it is used frequently as an explicit criterion on c. More refined conditions using Carleman estimates can be found in [**92**]. Bounds on ∇c and smoothness assumptions on c cannot be removed completely; if c is discontinuous, even on a single interface, localisation phenomena may occur, and the observability inequality fails, see [**149**]. In the same paper, counter-examples to observability are also provided for $c \in C^1$ with a large norm.

If we look at the delicate term in (2.10), namely

$$\int_{(0,T)\times\Omega} \left(\nabla(\log c) \cdot (x - x_0)\right) c^{-2} (\partial_t \varphi)^2 \, dt \, dx,$$

we see that it involves a combination of three quantities: a typical length-scale, represented by $x - x_0$; the gradient (and by extension the jump) of the velocity c in a given direction; and the time derivative of φ. Loosely speaking, the sufficient observability condition says that provided that the variations are not too large (in a good direction) compared to the domain size and the speed of propagation of the wave, observability reduces to the case of the constant coefficient equation. Thus, with respect to the variable velocity, Theorem 2.8 can be interpreted as a stability under small perturbation result.

2.3. On the relation with the Hilbert Uniqueness Method

In the previous section, we have established a setting in which Problem 2.1 (and more generally the inverse problem associated to (2.2)) is well-posed. If system (2.2) is observable at time T from Γ, consider the functional

$$(2.13) \qquad I(\varphi^a, \varphi^b) := \tfrac{1}{2} \int_0^T \int_\Gamma (\partial_\nu u - \partial_\nu \varphi)^2 \, \mathrm{d}\sigma \, \mathrm{d}t,$$

where $(\varphi^a, \varphi^b) \in H_0^1(\Omega) \times L^2(\Omega)$ and u is the solution of (2.2) with initial data (φ^a, φ^b), namely

$$(2.14) \qquad \begin{cases} c^{-2} \partial_{tt} u - \Delta u = 0 & \text{in } (0, T) \times \Omega, \\ u(0, \bullet) = \varphi^a & \text{in } \Omega, \\ \partial_t u(0, \bullet) = \varphi^b & \text{in } \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases}$$

By Lemma 2.4 and Theorem 2.8, we have

$$c^{-1} \big(\|\varphi_1 - \varphi^b\|_{L^2(\Omega)} + \|\varphi_0 - \varphi^a\|_{H_0^1(\Omega)} \big)$$
$$\leq I(\varphi^a, \varphi^b) \leq C \big(\|\varphi_1 - \varphi^b\|_{L^2(\Omega)} + \|\varphi_0 - \varphi^a\|_{H_0^1(\Omega)} \big)$$

for some positive constant C independent of $\varphi_0, \varphi_1, \varphi^a$ and φ^b. Thus, the determination of the initial conditions (φ_0, φ_1) of (2.2) may be performed by a minimisation procedure of the functional I. Introducing the adjoint problem

$$(2.15) \qquad \begin{cases} c^{-2} \partial_{tt} \psi - \Delta \psi = 0 & \text{in } (0, T) \times \Omega, \\ \psi(T, \bullet) = \psi_0 & \text{in } \Omega, \\ \partial_t \psi(T, \bullet) = \psi_1 & \text{in } \Omega, \\ \psi = v & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

we see that if $\psi_0 = \psi_1 = 0$ and $v = \mathbb{1}_\Gamma \partial_\nu \varphi$, we have [1], thanks to (2.14) and (2.15) and after an integration by parts,

$$-\int_0^T \int_\Gamma \partial_\nu \varphi \, \partial_\nu u \, d\sigma \, dt = -\left\langle c^{-2} \partial_t \psi(0, \bullet), \varphi^a \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}$$
$$+ \int_\Omega \psi(0, \bullet) c^{-2} \varphi^b \, dx.$$

Noting that

$$I(\varphi^a, \varphi^b) = \tfrac{1}{2} \int_0^T \int_\Gamma (\partial_\nu u)^2 \, d\sigma \, dt$$
$$- \int_0^T \int_\Gamma \partial_\nu \varphi \, \partial_\nu u \, d\sigma \, dt + \tfrac{1}{2} \int_0^T \int_\Gamma (\partial_\nu \varphi)^2 \, d\sigma \, dt,$$

as $\int_0^T \int_\Gamma (\partial_\nu \varphi)^2 \, d\sigma$ is fixed, we re-write the minimisation problem as

$$(2.16) \quad \min_{(\varphi^a, \varphi^b) \in H_0^1(\Omega) \times L^2(\Omega)} \tfrac{1}{2} \int_0^T \int_\Gamma (\partial_\nu u)^2 \, d\sigma \, dt$$
$$- \left\langle c^{-2} \partial_t \psi(0, \bullet), \varphi^a \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_\Omega \psi(0, \bullet) c^{-2} \varphi^b \, dx,$$

where $u \in L^2(0, T; H_0^1(\Omega))$ is the unique weak solution of (2.14) and ψ solves (2.15). This problem is precisely the minimisation problem appearing in the Hilbert Uniqueness Method of Lions [145], [146]. We refer to the extensive literature on that problem for effective numerical schemes and more details.

Note that, since $\mathbb{1}_\Gamma \partial_\nu \varphi \in L^2((0, T) \times \partial\Omega)$, problem (2.15) is set in a space that is too large for the classical theory to apply. In order to make sense of (2.15), we look for transposition (or dual) solutions.

---

1. Recall that the duality product is

$$\langle a, b \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega \nabla\big((-\Delta_0)^{-1} a\big) \cdot \nabla b \, dx,$$

where $\Delta_0 \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Laplace operator with homogeneous Dirichlet boundary conditions on Ω.

To this aim, we consider now

$$(2.17) \qquad \begin{cases} c^{-2} \partial_{tt}\varphi - \Delta\varphi = f & \text{in } (0,T) \times \Omega, \\ \varphi(0,\bullet) = 0 & \text{in } \Omega, \\ \partial_t \varphi(0,\bullet) = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

with $f \in L^1((0,T); L^2(\Omega))$ – and for the sake of brevity we will admit that Proposition 2.2 and Lemma 2.4 can be suitably adapted to problem (2.17).

Proposition 2.11. — *Let $c \in W^{1,\infty}(\Omega; \mathbb{R}_+)$ be such that $\log c \in W^{1,\infty}(\Omega)$ and $T > 0$.*

▷ *Given $\psi_0 \in L^2(\Omega)$, $\psi_1 \in H^{-1}(\Omega)$ and $v \in L^2((0,T) \times \partial\Omega)$, problem (2.15) has a unique solution $\psi \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$, defined in the sense of transposition.*

▷ *More precisely, for every $f \in L^1((0,T); L^2(\Omega))$ there holds*

$$\int_{(0,T)\times\Omega} \psi f \, dt \, dx - \int_\Omega \psi_0 c^{-2} \partial_t \varphi(T,\bullet) \, dx + \left\langle \psi_1, c^{-2}\varphi(T,\bullet) \right\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

$$= - \int_{(0,T)\times\partial\Omega} v \, \partial_\nu \varphi \, dt \, dx,$$

where φ is the solution of (2.17). Furthermore,

$$\|\psi\|_{L^\infty((0,T);L^2(\Omega))} + \|\partial_t \psi\|_{L^\infty((0,T);H^{-1}(\Omega))}$$

$$\leq C \left(\|v\|_{L^2((0,T)\times\partial\Omega)} + \|\psi_0\|_{L^2(\Omega)} + \|\psi_1\|_{H^{-1}(\Omega)} \right)$$

for some $C > 0$ depending only on Ω, T and $\|\log c\|_{W^{1,\infty}(\Omega)}$.

Remark 2.12. — In order to understand the reason for this definition, let us formally integrate by parts the differential equation satisfied by φ against ψ:

$$-\int_{(0,T)\times\Omega} f\psi\,dt\,dx = \int_0^T\int_\Omega (\Delta\varphi - c^{-2}\partial_{tt}\varphi)\psi\,dx\,dt$$

$$= \int_0^T\int_{\partial\Omega} (\partial_\nu\varphi\psi - \partial_\nu\psi\varphi)\,d\sigma\,dt + \int_0^T\int_\Omega \Delta\psi\varphi\,dx\,dt$$

$$+ \int_0^T\int_\Omega c^{-2}\partial_t\varphi\partial_t\psi\,dx\,dt + \int_0^T\int_\Omega \partial_t(-c^{-2}\partial_t\varphi\psi)\,dx\,dt.$$

Using the boundary conditions satisfied by ψ and φ and the differential equation satisfied by ψ, this implies

$$-\int_{(0,T)\times\Omega} f\psi\,dt\,dx = \int_{(0,T)\times\partial\Omega} \partial_\nu\varphi v\,dt\,d\sigma$$

$$+ \int_0^T\int_\Omega c^{-2}(\partial_{tt}\psi\varphi + \partial_t\varphi\partial_t\psi)\,dx\,dt - \int_\Omega [c^{-2}\psi\partial_t\varphi]_0^T\,dx$$

$$= \int_{(0,T)\times\partial\Omega} \partial_\nu\varphi v\,dt\,d\sigma + \int_\Omega c^{-2}[\varphi\partial_t\psi - \psi\partial_t\varphi]_0^T\,dx,$$

which is the identity we introduced in the definition of ψ.

Proof. — In this proof, C will change from line to line, and depend at most on Ω, T and $\|\log c\|_{W^{1,\infty}(\Omega)}$. Consider the map

$$L\colon L^1((0,T);L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$f \longmapsto \int_\Omega c^{-2}\psi_0\partial_t\varphi(T,\bullet)\,dx$$

$$- \int_{(0,T)\times\partial\Omega} v\partial_\nu\varphi\,dt\,d\sigma - \langle\psi_1, c^{-2}\varphi(T,\bullet)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

By (a variant of) Proposition 2.2, there holds

$$\int_\Omega c^{-2}(\partial_t\varphi(T,\bullet))^2 + |\nabla\varphi(T,\bullet)|^2\,dx \le C\|f\|_{L^1((0,T);L^2(\Omega))}^2.$$

Thus, in particular,

$$\left| \int_\Omega c^{-2} \psi_0 \, \partial_t \varphi(T, \bullet) \, \mathrm{d}x - \langle \psi_1, c^{-2} \varphi(T, \bullet) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \right|$$
$$\leq C \|f\|_{L^1((0,T); L^2(\Omega))} \big(\|\psi_0\|_{L^2(\Omega)} + \|\psi_1\|_{H^{-1}(\Omega)} \big).$$

By (a variant of) Lemma 2.4 we obtain

$$\int_{(0,T) \times \partial \Omega} v \, \partial_\nu \varphi \, \mathrm{d}t \, \mathrm{d}\sigma \leq C \|v\|_{L^2((0,T) \times \partial \Omega)} \cdot \|f\|_{L^1((0,T); L^2(\Omega))},$$

thus altogether

$$L(f) \leq C \Big(\|v\|_{L^2((0,T) \times \partial \Omega)} + \|\psi_0\|_{L^2(\Omega)} + \|\psi_1\|_{H^{-1}(\Omega)} \Big).$$

Hence, the Riesz representation theorem shows that there exists a unique ψ in $L^\infty((0,T); L^2(\Omega))$ such that

$$L(f) = \int_{(0,T) \times \Omega} f \psi \, \mathrm{d}t \, \mathrm{d}x,$$

which satisfies

$$\|\psi\|_{L^\infty((0,T); L^2(\Omega))} \leq C \Big(\|v\|_{L^2((0,T) \times \partial \Omega)} + \|\psi_0\|_{L^2(\Omega)} + \|\psi_1\|_{H^{-1}(\Omega)} \Big).$$

A density argument then shows that ψ belongs to

$$C\big([0,T]; L^2(\Omega)\big) \cap C^1\big([0,T]; H^{-1}(\Omega)\big).$$

Indeed, let $(v_n, \psi_{0,n}, \psi_{1,n}) \in C^\infty((0,T) \times \partial \Omega) \times C^\infty(\Omega)^2$ be such that

$$v_n \to v \ \text{ in } L^2((0,T) \times \partial \Omega), \ \ \psi_{0,n} \to \psi_0 \ \text{ in } L^2(\Omega), \ \ \psi_{1,n} \to \psi_1 \ \text{ in } H^{-1}(\Omega).$$

Then the solution ψ_n of (2.15) with data $(v_n, \psi_{0,n}, \psi_{1,n})$ is smooth and satisfies

$$\|\psi_n\|_{C([0,T]; L^2(\Omega))} \leq C \Big(\|v_n\|_{L^2((0,T) \times \partial \Omega)} + \|\psi_{0,n}\|_{L^2(\Omega)} + \|\psi_{1,n}\|_{H^{-1}(\Omega)} \Big).$$

By linearity, ψ_n is a Cauchy sequence: passing to the limit we obtain that ψ belongs to $C([0,T]; L^2(\Omega))$. The second estimate is similar. $\qquad\square$

Corollary 2.13. — *Let $c \in W^{1,\infty}(\Omega; \mathbb{R}_+)$ be such that $\log c \in W^{1,\infty}(\Omega)$ and $T > 0$. Given $v \in L^2((0,T) \times \partial \Omega)$, there exists a unique transposition*

solution $U \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ *of*

$$(2.18) \quad \begin{cases} c^{-2} \partial_{tt} U - \Delta U = 0 & in \ (0,T) \times \Omega, \\ U(T, \bullet) = 0 & in \ \Omega, \\ \partial_t U(T, \bullet) = 0 & in \ \Omega, \\ U = \mathbb{1}_\Gamma v & on \ (0,T) \times \partial\Omega, \end{cases}$$

given by Proposition 2.11. Furthermore, if p is the solution of (2.1), *there holds*

$$(2.19) \quad \int_{(0,T)\times\Gamma} \partial_\nu p v \, \mathrm{d}t \, \mathrm{d}\sigma = \langle c^{-2} \partial_t U(0, \bullet), A \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Remark 2.14. — Identity (2.19) shows that the dual solution U plays the role of a probe in practice: varying v and solving a direct problem to compute $\partial_t U(0, \bullet)$, we measure different moments of A.

Proof. — Proposition 2.11 does prove the existence of a unique transposition solution U, as problem (2.15) is of the same form as problem (2.18). Since U belongs to $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$, we may integrate it by parts against p, the weak solution of (2.1) in $(0,T) \times \Omega$. Formally, we have

$$0 = \int_{(0,T)\times\Omega} (\Delta U - c^{-2}\partial_{tt}U)p - (\Delta p - c^{-2}\partial_{tt}p)U \, \mathrm{d}t \, \mathrm{d}x$$

$$= \int_{(0,T)\times\partial\Omega} (\partial_\nu U p - U \partial_\nu p) \, \mathrm{d}t \, \mathrm{d}\sigma - . \int_\Omega c^{-2}\partial_t U p - \partial_t p U \, \mathrm{d}x \big|_0^T$$

$$= -\int_{(0,T)\times\Gamma} v \partial_\nu p \, \mathrm{d}t \, \mathrm{d}\sigma + \int_\Omega c^{-2}\partial_t U(0, \bullet) p \, \mathrm{d}x.$$

As all terms in the final identity are well defined when $\partial_\nu p, v \in L^2((0,T) \times \partial\Omega)$, $A \in H_0^1(\Omega)$ and $c^{-2}\partial_t U(0, \bullet) \in H^{-1}(\Omega)$, the conclusion is established by a density argument. $\qquad \square$

2.4. Proof of Lemma 2.4

The goal of this section is to provide a proof of the following lemma.

Lemma 2.4. — *Let $c \in W^{1,\infty}(\Omega)$ be such that $\min_{\overline{\Omega}} c > 0$, $T > 0$ and Γ be an open subset of $\partial\Omega$. The map*

$$(2.3) \qquad D_\Gamma : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow L^2\big((0,T) \times \Gamma\big), \quad (\varphi_0, \varphi_1) \longmapsto \partial_\nu \varphi,$$

where φ is the solution of (2.2)*, is continuous.*

Proof. — It is sufficient to consider the case $\Gamma = \partial\Omega$. As the boundary of Ω is C^2, there exists a function $h \in C^1(\overline{\Omega}; \mathbb{R}^d)$ such that $h = \nu$ on $\partial\Omega$ (see *e.g.* [**126**]). Testing (2.2) against $h \cdot \nabla\varphi$ formally, we obtain

$$(2.20) \qquad \int_{(0,T)\times\Omega} c^{-2} \partial_{tt}\varphi\, h \cdot \nabla\varphi + \nabla\varphi \cdot \nabla(h \cdot \nabla\varphi)\, \mathrm{d}t\,\mathrm{d}x$$

$$= \int_{(0,T)\times\partial\Omega} (\partial_\nu\varphi)^2\, \mathrm{d}t\,\mathrm{d}\sigma.$$

Write

$$I_1 = \int_{(0,T)\times\Omega} c^{-2} \partial_{tt}\varphi\, h \cdot \nabla\varphi\, \mathrm{d}t\,\mathrm{d}x, \quad I_2 = \int_{(0,T)\times\Omega} \nabla\varphi \cdot \nabla(h \cdot \nabla\varphi)\, \mathrm{d}t\,\mathrm{d}x,$$

and proceed to bound both integrals. We have

$$I_1 = \int_0^T \partial_t \int_\Omega c^{-2} \partial_t\varphi\, h \cdot \nabla\varphi\, \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega c^{-2} h \cdot \nabla\big(\tfrac{1}{2}(\partial_t\varphi)^2\big)\, \mathrm{d}x\,\mathrm{d}t$$

$$= \int_\Omega c^{-2} \partial_t\varphi\, h \cdot \nabla\varphi\, \mathrm{d}x\big|_0^T + \int_0^T \int_\Omega \mathrm{div}(c^{-2}h)\tfrac{1}{2}(\partial_t\varphi)^2\, \mathrm{d}x\,\mathrm{d}t.$$

Considering both terms on the right-hand side, we find

$$\left| \int_\Omega c^{-2} \partial_t\varphi\, h \cdot \nabla\varphi\, \mathrm{d}x \, \Big|_0^T \right| \le 4 \|c^{-1}\|_{L^\infty(\Omega)} \cdot \|h\|_{C^1(\overline{\Omega})} \sup_{t\in[0,T]} E(t),$$

$$\left| \int_{(0,T)\times\Omega} \mathrm{div}(c^{-2}h)\tfrac{1}{2}(\partial_t\varphi)^2\, \mathrm{d}t\,\mathrm{d}x \right| \le \big(2\|\nabla \log c\|_{L^\infty(\Omega)} + 1\big) \cdot \|h\|_{C^1(\overline{\Omega})} T \sup_{t\in[0,T]} E(t),$$

where we set $\|h\|_{C^1(\overline{\Omega})} = \|h\|_{L^\infty(\Omega)} + \|\nabla h\|_{L^\infty(\Omega)}$, and therefore

$$(2.21) \qquad |I_1| \le C\Big(\|c^{-1}\|_{L^\infty(\Omega)}, \|\nabla \log c\|_{L^\infty(\Omega)}, T, \|h\|_{C^1(\overline{\Omega})} \Big) \sup_{t\in[0,T]} E(t).$$

Let us now turn to I_2: expanding the integrand we find

$$I_2 = \int_{(0,T)\times\Omega} \partial_j\varphi\,\partial_j h_i\,\partial_i\varphi + \partial_i\varphi\, h_j\,\partial_{ij}\varphi\,\mathrm{d}t\,\mathrm{d}x,$$

where we have used Einstein's summation convention, according to which repeated indices (in this case both i and j) are implicitly summed over. The first term of the right-hand side is also bounded by the system's energy, since

$$\left|\int_{(0,T)\times\Omega} \partial_j\varphi\,\partial_j h_i\,\partial_i\varphi\,\mathrm{d}t\,\mathrm{d}x\right| \leq C(d)\|h\|_{C^1(\overline{\Omega})}T\sup_{t\in[0,T]} E(t).$$

As for the second term, integrating by parts once more we obtain

$$\int_{(0,T)\times\Omega} \partial_i\varphi\, h_j\,\partial_{ij}\varphi\,\mathrm{d}t\,\mathrm{d}x = -\frac{1}{2}\int_{(0,T)\times\Omega}\mathrm{div}(h)|\nabla\varphi|^2\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{2}\int_0^T\int_{\partial\Omega}(\partial_\nu\varphi)^2\,\mathrm{d}\sigma\,\mathrm{d}t,$$

therefore

$$(2.22)\qquad \left|I_2 - \frac{1}{2}\int_0^T\int_{\partial\Omega}(\partial_\nu\varphi)^2\,\mathrm{d}\sigma\,\mathrm{d}t\right| \leq C\big(d,T,\|h\|_{C^1(\overline{\Omega})}\big)\sup_{t\in[0,T]} E(t).$$

Combining (2.20), (2.21) and (2.22), we obtain

$$\frac{1}{2}\int_{(0,T)\times\partial\Omega}|\partial_\nu\varphi|^2\,\mathrm{d}t\,\mathrm{d}\sigma \leq C\big(d,\|\log c\|_{W^{1,\infty}(\Omega)},T,\|h\|_{C^1(\overline{\Omega})}\big)\sup_{t\in[0,T]} E(t),$$

as announced. $\square$

CHAPTER 3

REGULARITY THEORY FOR MAXWELL'S EQUATIONS

3.1. Introduction

The focus of this chapter is the regularity of weak solutions to the time harmonic Maxwell equations

$$(3.1) \qquad \begin{cases} \operatorname{curl} E = i\omega\mu H + K & \text{in } \Omega, \\ \operatorname{curl} H = -i\gamma E + J & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain, $\omega \in \mathbb{C}$ is the frequency, μ and γ are the electromagnetic parameters in $L^\infty(\Omega; \mathbb{C}^{3\times3})$, K and J are the current sources in $L^2(\Omega; \mathbb{C}^3)$ and the weak solutions $E, H \in H(\operatorname{curl}, \Omega)$ are the electric and magnetic fields, respectively, where

$$H(\operatorname{curl}, \Omega) := \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3)\}.$$

In other words, E and H only have a well-defined curl, but not a full gradient. A natural regularity question is whether E and H have full weak derivatives in L^2, namely $E, H \in H^1$. This step is unnecessary for second-order elliptic equations in divergence form, as it is implicit in the weak formulation.

The second question we would like to address is the Hölder continuity of the solutions. This is a classical topic for elliptic equations, thanks to the De Giorgi-Nash-Moser theorem and to the Schauder estimates for the Hölder continuity

of the derivatives. The continuity of the solutions is of importance to us, as internal data have to be interpreted pointwise.

Without further smoothness assumptions on the coefficients, the solutions need to be H^1, nor Hölder continuous. We focus on low (and sometimes optimal) additional regularity assumptions, as the electromagnetic parameters may not be smooth in practice. If the coefficients are isotropic and constant, smoothness of the solutions follows from the following inequality due to Friedrichs [**97**], [**106**]

$$\|u\|_{H^1(\Omega)} \le C\left(\|\operatorname{div} u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{L^2(\Omega)} + \|u \times \nu\|_{H^{1/2}(\partial\Omega)}\right).$$

The H^1 regularity of electromagnetic fields for anisotropic Lipschitz coefficients was considered in [**210**], and Hölder regularity with isotropic complex Lipschitz coefficients was shown in [**217**]. Both papers make use of the scalar and vector potentials of the electric and magnetic fields. A different approach based on a different formulation of (3.1) in terms of a coupled elliptic system and on the L^p theory for elliptic equations was considered in [**12**], where H^1 and Hölder regularity was proved with complex anisotropic, possibly non symmetric, $W^{1,3+\delta}$ coefficients.

In this chapter, we present a method that combines these two approaches. Namely, we apply the L^p elliptic theory to the equations satisfied by the scalar and vector potentials. We show that H^1 regularity is granted with $W^{1,3}$ coefficients whereas Hölder regularity always holds provided the coefficients themselves are Hölder continuous. This last result was proved in [**8**] and is optimal. Without additional regularity assumptions on the coefficients, our approach allows for a proof of higher integrability properties for the fields, thanks to the Gehring's lemma.

In order to understand why these results and the corresponding assumptions are natural, it is instructive to consider the case when $\omega = 0$ and $K \equiv 0$. Since $\gamma = \omega\varepsilon + i\sigma$, for $\omega = 0$ system (3.1) simply reduces to the conductivity equation

$$-\operatorname{div}(\sigma\nabla q) = \operatorname{div} J \quad \text{in} \ \Omega, \quad E = \nabla q \quad \text{in} \ \Omega.$$

In this case, the H^1 and $C^{0,\alpha}$ regularity for E corresponds to the H^2 and $C^{1,\alpha}$ regularity for the scalar potential q, respectively. In view of classical elliptic

regularity results (in particular, the L^p theory for elliptic equations with VMO coefficients and standard Schauder estimates), $q \in H^2$ if $\sigma \in W^{1,3}$ and $q \in C^{1,\alpha}$ provided that $\sigma \in C^{0,\alpha}$, with *ad hoc* assumptions on the source J.

The aim of this chapter is to show that this argument may be extended to the general case, for any frequency $\omega \in \mathbb{C}$. As mentioned above, this is achieved by using the Helmholtz decomposition, namely

$$E = \nabla q + \operatorname{curl} \Phi.$$

We show that the vector potential Φ is always more regular that the scalar potential q. This allows us to reduce the problem to a regularity analysis for q, exactly as above in the case $\omega = 0$, by using the elliptic PDE satisfied by q. This argument is applied simultaneously for E and H.

For simplicity, only interior (local) regularity will be discussed in this work; global regularity may be obtained by a careful analysis of the boundary conditions [12], [8]. We will focus on the case when μ and γ enjoy the same regularity. The general case is more involved, and can be addressed using Campanato estimates [217], [12], [8].

This chapter is structured as follows. In section 3.2 we discuss some preliminary results on elliptic regularity theory. Section 3.3 contains the main regularity theorems for Maxwell's system. The results on the H^1 and $W^{1,p}$ regularity are new. Further, as far as the authors are aware, the higher integrability result for the electric and magnetic fields (a consequence of Gehring's lemma) has not been reported in the literature before. Even though the focus of this chapter is regularity of weak solutions, in Section 3.4 we recall classical results on well-posedness for the Dirichlet problem associated to (3.1) for the sake of completeness.

3.2. Preliminaries

The theory for second order elliptic equations is completely established in the Hilbert case, namely for $p = 2$. For a uniformly elliptic tensor μ, the divergence form equation

$$- \operatorname{div}(\mu \nabla u) = \operatorname{div} F \quad \text{in } \Omega$$

admits a unique solution in $W_0^{1,2}(\Omega;\mathbb{C})$ for a fixed $F \in L^2(\Omega;\mathbb{C}^3)$: this simply follows by the Lax–Milgram theorem. For $p > 2$, whether $F \in L^p$ implies $u \in W^{1,p}$ depends on the regularity of μ. Without further assumptions on μ, this is not the case. Continuity of μ is sufficient [**197**], but not necessary. The weaker assumption $\mu \in \mathrm{VMO}(\Omega)$ is sufficient, where the space VMO consists of functions with vanishing mean oscillations, namely of those functions f such that

$$\lim_{|Q| \to 0} \frac{1}{|Q|} \int_Q \left| f - \frac{1}{|Q|} \int f \, \mathrm{d}t \right| \mathrm{d}x = 0$$

for all cubes Q. In these notes, we only use that $W^{1,d}(\Omega)$ and $C(\Omega)$ are continuously embedded in VMO, where d is the dimension [**62**]. In this chapter, $d = 3$.

Lemma 3.1 (see [37]). — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain and take $\Omega' \Subset \Omega$. Let $\mu \in \mathrm{VMO}(\Omega;\mathbb{C}^{3\times3})$ be such that*

$$(3.2) \qquad 2\Lambda^{-1}|\zeta|^2 \le \zeta \cdot (\mu + \overline{\mu}^T)\zeta, \quad |\mu| \le \Lambda \quad a.e. \text{ in } \Omega.$$

Take $F \in L^p(\Omega;\mathbb{C}^3)$ for some $p \in [2,\infty)$ and let $u \in H^1(\Omega;\mathbb{C})$ be a weak solution of

$$-\operatorname{div}(\mu \nabla u) = \operatorname{div} F \quad in \ \Omega.$$

Then $u \in W_{\mathrm{loc}}^{1,p}(\Omega;\mathbb{C})$ and

$$\|u\|_{W^{1,p}(\Omega')} \le C\big(\|u\|_{H^1(\Omega)} + \|F\|_{L^p(\Omega)}\big)$$

for some $C > 0$ depending only on Ω, Ω', Λ and $\|\mu\|_{\mathrm{VMO}(\Omega)}$.

We shall also need an H^2 regularity result for elliptic equations. The standard formulation given in many textbooks on PDE [**95**], [**103**], [**104**], [**206**] requires Lipschitz coefficients. Using Lemma 3.1, we provide here an improved version, assuming $W^{1,3}$ regularity for the coefficients.

Lemma 3.2. — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain and take $\Omega' \Subset \Omega$. Given $\mu \in W^{1,3}(\Omega;\mathbb{C}^{3\times3})$ satisfying (3.2) and $f \in L^2(\Omega;\mathbb{C})$, let $u \in H^1(\Omega;\mathbb{C})$ be a weak solution of*

$$-\operatorname{div}(\mu \nabla u) = f \quad in \ \Omega.$$

Then $u \in H^2_{\mathrm{loc}}(\Omega; \mathbb{C})$ and

$$\|u\|_{H^2(\Omega')} \leq C \big(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \big)$$

for some $C > 0$ depending only on Ω, Ω', Λ and $\|\mu\|_{W^{1,3}(\Omega)}$.

Proof. — For simplicity, we prove the result in the simpler case of isotropic coefficient and by using the strong form of the equation. The general case can be proved substantially in the same way, by passing to the weak formulation and using the standard difference quotient method [**95**], [**103**]. In this proof we write C for any positive constant depending on Ω, Ω', Λ and $\|\mu\|_{W^{1,3}(\Omega)}$ only.

Let Ω'' be a smooth subdomain such that $\Omega' \Subset \Omega'' \Subset \Omega$ and $v \in H^1_0(\Omega; \mathbb{C})$ satisfy $\Delta v = f$. Without loss of generality, assume that Ω is smooth. Indeed, if Ω is not smooth, introduce another smooth intermediate subdomain Ω''', such that $\Omega'' \Subset \Omega''' \Subset \Omega$, in lieu of Ω in what follows. Standard H^2 estimates for elliptic equations [**105**, Theorem 8.12] give $v \in H^2(\Omega; \mathbb{C})$ with $\|v\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$. Thus, the Sobolev embedding theorem yields

$$(3.3) \qquad \|v\|_{W^{1,6}(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

The equation for u becomes $-\operatorname{div}(\mu \nabla u) = \operatorname{div}(\nabla v)$ in Ω. In view of Lemma 3.1 we have $u \in W^{1,6}_{\mathrm{loc}}(\Omega; \mathbb{C})$ and

$$\|u\|_{W^{1,6}(\Omega'')} \leq C \big(\|u\|_{H^1(\Omega)} + \|\nabla v\|_{L^6(\Omega)} \big).$$

Hence by (3.3) we obtain

$$(3.4) \qquad \|u\|_{W^{1,6}(\Omega'')} \leq C \big(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} \big).$$

Next, note that the equation for u can be restated as

$$-\Delta u = \mu^{-1} \nabla \mu \cdot \nabla u + \mu^{-1} f \quad \text{in } \Omega''.$$

Applying again standard H^2 estimates we obtain that $u \in H^2(\Omega'; \mathbb{C})$ and

$$\|u\|_{H^2(\Omega')} \leq C \big(\|u\|_{H^1(\Omega'')} + \|\mu^{-1} \nabla \mu \cdot \nabla u\|_{L^2(\Omega'')} + \|\mu^{-1} f\|_{L^2(\Omega'')} \big)$$
$$\leq C \big(\|u\|_{H^1(\Omega)} + \|\mu^{-1} \nabla \mu\|_{L^3(\Omega'')} \|\nabla u\|_{L^6(\Omega'')} + \|f\|_{L^2(\Omega'')} \big)$$
$$\leq C \big(\|u\|_{H^1(\Omega)} + \|u\|_{W^{1,6}(\Omega'')} + \|f\|_{L^2(\Omega'')} \big).$$

Inserting (3.4) in this last estimate provides the result. $\qquad \square$

As it is central to our argument, we remind the reader of a fundamental tool, known as Meyers' theorem [**157**], which we choose to present as a consequence of Gehring's lemma [**100**]; see also [**61**]. The following quantitative version is proved in [**118**].

Lemma 3.3 (Gehring's lemma). — *Let $\Omega = Q_0$ be a cube. Given $1 < p < \infty$, let $w, g \in L^p(\Omega)$ be non-negative functions and $C > 0$ such that for all cubes Q such that $Q \subset 2Q \subset \Omega$, there holds*

$$\left(\fint_Q w^p \right)^{1/p} \leq C \left(\fint_{2Q} w \right) + \left(\fint_{2Q} g^p \right)^{1/p}.$$

Then, for each $0 < \tau < 1$ and $p < s < p + \frac{p-1}{10^{d+p} 4^p C^p}$ we have

$$\left(\fint_{\tau Q_0} w^s \right)^{1/s} \leq \frac{10^{2d}}{\tau^{d/s}(1-\tau)^{d/p}} \left[\left(\fint_{Q_0} w^p \right)^{1/p} + \left(\fint_{Q_0} g^s \right)^{1/s} \right].$$

The following corollary follows from a covering argument.

Corollary 3.4. — *Let Ω be a bounded connected open set in $\mathbb{R}^3$. Given $1 < p < \infty$, let $w, g \in L^p(\Omega)$ be non-negative functions and $C > 0$ such that for all cubes Q such that $Q \subset 2Q \subset \Omega$, there holds*

$$\left(\fint_Q w^p \right)^{1/p} \leq C \left(\fint_{2Q} w \right) + \left(\fint_{2Q} g^p \right)^{1/p}.$$

Then, for each $\Omega' \Subset \Omega$ and each $p < s < p + \frac{p-1}{10^{d+p} 4^p C^p}$ we have

$$\left(\int_{\Omega'} w^s \right)^{1/s} \leq C(\Omega, \Omega', s, p) \left[\left(\int_{\Omega} w^p \right)^{1/p} + \left(\int_{\Omega} g^s \right)^{1/s} \right].$$

This result implies local higher integrability estimates for solutions of second order elliptic systems with heterogeneous coefficients. We give below one such estimate which is sufficient for our purposes.

Theorem 3.5 (Meyers' theorem). — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Let A in $L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ be a symmetric matrix such that for every $\zeta \in \mathbb{R}^3$*

$$(3.5) \qquad \Lambda^{-1} |\zeta|^2 \leq \zeta \cdot \left(\mathrm{Re}(A(x)) \zeta \right) \leq \Lambda |\zeta|^2 \quad \text{almost everywhere in } \Omega$$

for some $\Lambda > 0$. *Given* $q > 2$, $g \in L^q(\Omega; \mathbb{C}^3)$ *and* $f \in L^p(\Omega; \mathbb{C}^3)$ *with* $p = \frac{3q}{q+3}$, *let* $u \in H^1(\Omega; \mathbb{C})$ *be a weak solution of*

$$(3.6) \qquad -\operatorname{div}(A\nabla u) = -\operatorname{div}(g) + f \quad in \ \Omega.$$

Then there exists $s > 0$ *depending only on* Ω *and* Λ *such that if* $q \geq s + 2$ *then* $\nabla u \in L^s_{\mathrm{loc}}(\Omega; \mathbb{C}^3)$ *and for each* $\Omega' \Subset \Omega$ *there holds*

$$\|\nabla u\|_{L^{2+s}(\Omega')} \leq C(\Omega, \Omega', \Lambda)\big(\|u\|_{H^1(\Omega)} + \|g\|_{L^q(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

Proof. — Note that we may assume without loss of generality that $f \equiv 0$. Indeed, let B be an open ball containing $\overline{\Omega}$, of radius the diameter of Ω. Extend f by zero outside Ω, and define $\psi_f \in H^1_0(B; \mathbb{C})$ as the unique solution of

$$-\Delta \psi_f = f \quad in \ B.$$

Applying Lemma 3.1 to the partial derivatives of f, we find $\psi_f \in W^{2,p}_{\mathrm{loc}}(B; \mathbb{C})$, and

$$\|\nabla \psi_f\|_{W^{1,p}(\Omega)} \leq C(\Omega)\|f\|_{L^p(B)} = C(\Omega)\|f\|_{L^p(\Omega)}.$$

Thanks to the Sobolev embedding theorem, it follows that $\nabla \psi_f \in L^q(\Omega; \mathbb{C}^3)$. We may therefore assume $f \equiv 0$, replacing g by $g + \nabla \psi_f$.

If we integrate (3.6) against $\bar{u}\chi^2$, where $0 \leq \chi \leq 1$ is a smooth compactly supported function in Ω, we obtain

$$(3.7) \qquad \operatorname{Re}\big\langle -\operatorname{div}(A\nabla u), \bar{u}\chi^2 \big\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \leq \Big|\big\langle -\operatorname{div}(g), \bar{u}\chi^2 \big\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}\Big|,$$

and

$$\operatorname{Re}\big\langle -\operatorname{div}(A\nabla u), \bar{u}\chi^2 \big\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega \operatorname{Re}(A)\nabla(u\chi)\cdot\overline{\nabla(u\chi)} - \int_\Omega \operatorname{Re}(A)\nabla\chi\cdot\nabla\chi |u|^2.$$

This implies, thanks to (3.5),

$$(3.8) \qquad \operatorname{Re}\big\langle -\operatorname{div}(A\nabla u), \bar{u}\chi^2 \big\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$

$$\geq \Lambda^{-1}\|\nabla(u\chi)\|^2_{L^2(\Omega)} - \Lambda\|\nabla\chi\|^2_{L^\infty(\Omega)} \cdot \|\mathbb{1}_\chi u\|^2_{L^2(\Omega)},$$

where $\mathbb{1}_\chi := \mathbb{1}_{\mathrm{supp}\,\chi}$ denotes the characteristic function of the support of χ.

Applying Young's inequality, on the right-hand side of (3.7) we obtain

$$
(3.9) \qquad \left| \langle - \operatorname{div}(g), \bar{u}\chi^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right|
$$
$$
\leq \frac{1}{2\Lambda} \| \nabla(u\chi) \|_{L^2(\Omega)}^2 + \tfrac{1}{2} \| \nabla\chi \|_{L^\infty(\Omega)}^2 \cdot \| u\chi \|_{L^2(\Omega)}^2
$$
$$
+ \tfrac{1}{2}(\Lambda + 1) \| \mathbb{1}_\chi g \|_{L^2(\Omega)}^2.
$$

Using the lower bound (3.8) and the upper bound (3.9) in inequality (3.7) we obtain

$$
\| \nabla(u\chi) \|_{L^2(\Omega)}^2 \leq C(\Lambda) \left(\| \nabla\chi \|_{L^\infty(\Omega)}^2 \cdot \| \mathbb{1}_\chi u \|_{L^2(\Omega)}^2 + \| \mathbb{1}_\chi g \|_{L^2(\Omega)}^2 \right).
$$

Given any cube Q in Ω such that $Q \subset 2Q \subset \Omega$, let d_Q be the length of its edges. Take χ to be such that $\chi \equiv 1$ on Q, supported on $2Q$, and such that $d_Q \| \nabla\chi \|_{L^\infty(\Omega)}$ is bounded by a universal constant. Note that we can safely replace u by $u - \fint_{2Q} u$ in the above inequality. Then, we have shown

$$
\| \nabla u \|_{L^2(Q)}^2 \leq C(\Lambda) \left(\frac{1}{d_Q^2} \left\| u - \fint_{2Q} u \right\|_{L^2(2Q)}^2 + \| g \|_{L^2(2Q)}^2 \right).
$$

Recall that $W^{1,6/5}(2Q)$ is continuously embedded in $L^2(2Q)$, and

$$
\left\| u - \fint_{2Q} u \right\|_{L^2(2Q)} \leq d_Q C \| \nabla u \|_{L^{6/5}(2Q)}, \quad u \in W^{1,6/5}(2Q),
$$

where C is a universal constant; the dependence on d_Q follows from a scaling argument [2]. Writing $w = |\nabla u|^{\frac{6}{5}}$ we have obtained

$$
\fint_Q w^{5/3} \leq C(\Lambda) \left[\left(\fint_{2Q} w \right)^{5/3} + \fint_{2Q} \left(g^{6/5} \right)^{5/3} \right]
$$

which in turns implies

$$
\left(\fint_Q w^r \right)^{1/r} \leq C(\Lambda) \left[\fint_{2Q} w + \left(\fint_{2Q} \left(g^{6/5} \right)^r \right)^{1/r} \right],
$$

with $r = \frac{5}{3}$. The result now follows from Corollary 3.4. $\qquad \square$

---

2. This inequality is the usual Poincaré-Sobolev inequality in the unit cube Q_1 centred at the origin, and $u \in W^{1,6/5}(2Q)$ can be written $u(x) = v((x - x_0)/d_Q)$ with $v \in W^{1,6/5}(Q_1)$, where x_0 is the centre of Q.

We also use the following estimate. Even though only the case $p = 2$ will be used in these notes, we state a general version for completeness.

Lemma 3.6 (see [106], [34]). — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$. Take $p \in (1, \infty)$ and $F \in L^p(\Omega; \mathbb{C}^3)$ such that $\operatorname{curl} F \in L^p(\Omega; \mathbb{C}^3)$, $\operatorname{div} F \in L^p(\Omega; \mathbb{C})$ and either $F \cdot \nu = 0$ or $F \times \nu = 0$ on $\partial\Omega$. Then $F \in W^{1,p}(\Omega; \mathbb{C}^3)$ and*

$$\|F\|_{W^{1,p}(\Omega)} \leq C\big(\|\operatorname{curl} F\|_{L^p(\Omega)} + \|\operatorname{div} F\|_{L^p(\Omega)}\big)$$

for some $C > 0$ depending only on Ω and p.

The last preliminary lemma we need is the Helmholtz decomposition for L^2 vector fields.

Lemma 3.7 (see [34, Theorem 6.1], [33, Section 3.5]). — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$ and take $F \in L^2(\Omega; \mathbb{C}^3)$.*

1) *There exist $q \in H_0^1(\Omega; \mathbb{C})$ and $\Phi \in H^1(\Omega; \mathbb{C}^3)$ such that*

$$F = \nabla q + \operatorname{curl} \Phi \quad in \ \Omega,$$

   $\operatorname{div} \Phi = 0$ *in Ω and $\Phi \cdot \nu = 0$ on $\partial\Omega$.*
2) *There exist $q \in H^1(\Omega; \mathbb{C})$ and $\Phi \in H^1(\Omega; \mathbb{C}^3)$ such that*

$$F = \nabla q + \operatorname{curl} \Phi \quad in \ \Omega,$$

   $\operatorname{div} \Phi = 0$ *in Ω and $\Phi \times \nu = 0$ on $\partial\Omega$.*

In particular, we have $-\Delta\Phi = \operatorname{curl} F$ in Ω. In both cases, there exists $C > 0$ depending only on Ω such that

$$\|\Phi\|_{H^1(\Omega)} + \|\nabla q\|_{L^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}.$$

When applied to electromagnetic fields, this decomposition leads to the following systems.

Corollary 3.8. — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$. Let $E, H \in H(\operatorname{curl}, \Omega)$ be weak solutions of Maxwell's system (3.1). There exist $q_E \in H_0^1(\Omega; \mathbb{C})$, $q_H \in H^1(\Omega; \mathbb{C})$ and*

$\Phi_E, \Phi_H \in H^1(\Omega; \mathbb{C}^3)$ *such that*

$$(3.10) \qquad E = \nabla q_E + \operatorname{curl} \Phi_E, \quad H = \nabla q_H + \operatorname{curl} \Phi_H,$$

and

$$(3.11) \qquad \begin{cases} -\Delta \Phi_E = i\omega\mu H + K & in \ \Omega, \\ \operatorname{div} \Phi_E = 0 & in \ \Omega, \\ \Phi_E \cdot \nu = 0 & on \ \partial\Omega, \end{cases} \quad \begin{cases} -\Delta \Phi_H = -i\gamma E + J & in \ \Omega, \\ \operatorname{div} \Phi_H = 0 & in \ \Omega, \\ \Phi_H \times \nu = 0 & on \ \partial\Omega. \end{cases}$$

Moreover, there exists $C > 0$ depending only on Ω such that

$$(3.12) \qquad \left\| (\Phi_E, \Phi_H) \right\|_{H^1(\Omega)^2} + \left\| (\nabla q_E, \nabla q_H) \right\|_{L^2(\Omega)^2} \leq C \left\| (E, H) \right\|_{L^2(\Omega)^2}.$$

We have now collected all the necessary ingredients to state and prove the main results of this chapter.

3.3. The main results

We consider weak solutions $E, H \in H(\operatorname{curl}, \Omega)$ to

$$(3.13) \qquad \begin{cases} \operatorname{curl} E = i\omega\mu H + K & in \ \Omega, \\ \operatorname{curl} H = -i\gamma E + J & in \ \Omega. \end{cases}$$

Throughout this section, we make the following assumptions:

▷ $\Omega \subseteq \mathbb{R}^3$ is a bounded simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$;

▷ the frequency ω belongs to $\mathbb{C}$;

▷ $\gamma = \omega\varepsilon + i\sigma$ is the admittivity of the medium, $\mu \in L^\infty(\Omega; \mathbb{C}^{3\times3})$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3\times3})$ are the magnetic permeability, electric permittivity and conductivity, respectively. We will assume that they satisfy suitable uniform ellipticity properties, namely there exists $\Lambda > 0$ such that for all $\zeta \in \mathbb{R}^3$

$$(3.14) \qquad \begin{cases} 2\Lambda^{-1}|\zeta|^2 \leq \zeta \cdot (\mu + \overline{\mu}^T)\zeta, \\ \Lambda^{-1}|\zeta|^2 \leq \zeta \cdot \varepsilon\zeta, \\ |\mu| + |\varepsilon| + |\sigma| \leq \Lambda \quad \text{a.e. in } \Omega; \end{cases}$$

▷ in the case $\omega = 0$, we also assume that for every $\zeta \in \mathbb{R}^3$

$$(3.15) \qquad \Lambda^{-1}|\zeta|^2 \leq \zeta \cdot \sigma\zeta \qquad \text{a.e. in } \Omega,$$

that $\operatorname{div} K = 0$ in Ω and that the following equation holds:

$$(3.16) \qquad\qquad -\operatorname{div}(\mu H) = 0 \quad \text{in} \ \ \Omega.$$

It is worth observing that the topological assumptions on Ω are not restrictive, since regularity is a local property. In the following, for $p \in (1, \infty)$ we shall make use of the space

$$W^{1,p}(\operatorname{div}, \Omega) := \big\{ F \in L^p(\Omega; \mathbb{C}^3) : \operatorname{div} F \in L^p(\Omega; \mathbb{C}) \big\},$$

equipped with the canonical norm. For $p = 2$, we set

$$H(\operatorname{div}, \Omega) := W^{1,2}(\operatorname{div}, \Omega).$$

We start with the interior H^1 regularity result.

Theorem 3.9. — *Take* $\Omega' \Subset \Omega$ *and* $\omega \in \mathbb{C}$. *Let*

$$\mu \in W^{1,3}(\Omega; \mathbb{C}^{3\times3}) \quad \text{and} \quad \varepsilon, \sigma \in W^{1,3}(\Omega; \mathbb{R}^{3\times3})$$

satisfy (3.14) *(and* (3.15) *if* $\omega = 0$). *Take* $J, K \in H(\operatorname{div}, \Omega)$ *(with* $\operatorname{div} K = 0$ *if* $\omega = 0$). *Let* $(E, H) \in H(\operatorname{curl}, \Omega)^2$ *be a weak solution of* (3.13) *(augmented with* (3.16) *if* $\omega = 0$). *Then* $(E, H) \in H^1_{\operatorname{loc}}(\Omega; \mathbb{C}^3)^2$ *and*

$$\big\| (E, H) \big\|_{H^1(\Omega')^2} \leq C \Big(\big\| (E, H) \big\|_{L^2(\Omega)^2} + \|J\|_{H(\operatorname{div}, \Omega)} + \|K\|_{H(\operatorname{div}, \Omega)} \Big)$$

for some $C > 0$ *depending only on* Ω, Ω', Λ, $|\omega|$ *and* $\|(\mu, \gamma)\|_{W^{1,3}(\Omega)^2}$.

Proof. — With an abuse of notation, in the proof several positive constants depending only on Ω, Ω', Λ, $|\omega|$ and $\|(\mu, \gamma)\|_{W^{1,3}(\Omega)^2}$ will be denoted by C.

Let Ω'' be a smooth domain such that $\Omega' \Subset \Omega'' \Subset \Omega$. Using the decompositions (3.10) given in Corollary 3.8 and applying Lemma 3.2 to (3.11) we obtain

$$(3.17) \qquad \big\| (\Phi_E, \Phi_H) \big\|_{H^2(\Omega'')^2} \leq C \big(\big\| (E, H) \big\|_{L^2(\Omega)^2} + \big\| (J, K) \big\|_{L^2(\Omega)^2} \big),$$

which, by the Sobolev embedding theorem, yields

$$(3.18) \qquad \big\| (\Phi_E, \Phi_H) \big\|_{W^{1,6}(\Omega'')^2} \leq C \big(\big\| (E, H) \big\|_{L^2(\Omega)^2} + \big\| (J, K) \big\|_{L^2(\Omega)^2} \big).$$

Taking the divergence of the equations in (3.13) and inserting the decompositions (3.10) yields

$$(3.19) \qquad \begin{cases} -\operatorname{div}(\mu\nabla q_H) = \operatorname{div}(\mu\operatorname{curl}\Phi_H - \mathrm{i}\omega^{-1}K) & \text{in } \Omega, \\ -\operatorname{div}(\gamma\nabla q_E) = \operatorname{div}(\gamma\operatorname{curl}\Phi_E + \mathrm{i}J) & \text{in } \Omega. \end{cases}$$

(In the case $\omega = 0$, by (3.16) the system reads

$$(3.20) \qquad \begin{cases} -\operatorname{div}(\mu\nabla q_H) = \operatorname{div}(\mu\operatorname{curl}\Phi_H) & \text{in } \Omega, \\ -\operatorname{div}(\sigma\nabla q_E) = \operatorname{div}(\sigma\operatorname{curl}\Phi_E + J) & \text{in } \Omega, \end{cases}$$

and the same argument given below applies.) To prove our claim, we must exhibit an H^2 estimate of q_H and q_E. Expanding the right-hand sides of the above equations, and using Einstein summation convention, which is that repeated indices are implicitly summed over, we obtain

$$-\operatorname{div}(\mu\nabla q_H) = \partial_i\mu_{ij}(\operatorname{curl}\Phi_H)_j + \mu_{ij}\,\partial_i(\operatorname{curl}\Phi_H)_j - \mathrm{i}\omega^{-1}\operatorname{div}K \quad \text{in } \Omega'',$$

$$-\operatorname{div}(\gamma\nabla q_E) = \partial_i\gamma_{ij}(\operatorname{curl}\Phi_E)_j + \gamma_{ij}\,\partial_i(\operatorname{curl}\Phi_E)_j + \mathrm{i}\operatorname{div}J \qquad \text{in } \Omega''.$$

Applying Lemma 3.2 and the Sobolev embedding theorem we obtain $q_E, q_H \in H^2_{\mathrm{loc}}(\Omega;\mathbb{C})$ and

$$\big\|(q_E, q_H)\big\|_{H^2(\Omega')^2} \le C\Big(\|(q_E, q_H)\|_{H^1(\Omega)^2} + \|(\Phi_E, \Phi_H)\|_{H^2(\Omega'')^2} + \|(J, K)\|_{H(\operatorname{div},\Omega)^2}\Big).$$

Thanks to (3.12) and (3.17) we have

$$\big\|(q_E, q_H)\big\|_{H^2(\Omega')} \le C\big(\|(E, H)\|_{L^2(\Omega)^2} + \|(J, K)\|_{H(\operatorname{div},\Omega)^2}\big).$$

The conclusion follows by combining the last inequality with (3.10) and (3.17). □

The following result provides local $C^{0,\alpha}$ estimates, see [**8**].

Theorem 3.10. — *Take $\alpha \in (0, 1/2]$, $\Omega' \Subset \Omega$ and $\omega \in \mathbb{C}$. Let*

$$\mu \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}) \quad \text{and} \quad \varepsilon, \sigma \in C^{0,\alpha}(\overline{\Omega};\mathbb{R}^{3\times 3})$$

satisfy (3.14) (and (3.15) if $\omega = 0$). Take $J, K \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ (with $\operatorname{div}K = 0$ if $\omega = 0$). Let $(E, H) \in H(\operatorname{curl},\Omega)^2$ be a weak solution of (3.13) (augmented with (3.16) if $\omega = 0$). Then $(E, H) \in C^{0,\alpha}(\Omega;\mathbb{C}^3)^2$ and

$$\big\|(E, H)\big\|_{C^{0,\alpha}(\overline{\Omega'})^2} \le C\big(\|(E, H)\|_{L^2(\Omega)^2} + \|(J, K)\|_{C^{0,\alpha}(\overline{\Omega})^2}\big)$$

for some $C > 0$ depending only on Ω, Ω', Λ, $|\omega|$ and $\|(\mu, \gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$.

Proof. — By (3.18), we have that $\operatorname{curl}\Phi_E, \operatorname{curl}\Phi_H \in L^6_{\mathrm{loc}}(\Omega;\mathbb{C}^3)$. Therefore, by Lemma 3.1 applied to (3.19) ((3.20) if $\omega = 0$) we obtain that $\nabla q_E, \nabla q_H$ belongs to $L^6_{\mathrm{loc}}(\Omega;\mathbb{C}^3)$, therefore thanks to (3.10) we have $E, H \in L^6_{\mathrm{loc}}(\Omega;\mathbb{C}^3)$. Differentiating the systems (3.11) we obtain for every $i = 1, 2, 3$

$$(3.21) \qquad -\Delta(\partial_i \Phi_E) = \partial_i(i\omega\mu H + K), \quad -\Delta(\partial_i \Phi_H) = \partial_i(-i\gamma E + J).$$

Thus, Lemma 3.1 yields $\Phi_E, \Phi_H \in W^{2,6}_{\mathrm{loc}}(\Omega)$. By the Sobolev embedding theorem, this implies $\Phi_E, \Phi_H \in C^{1,1/2}(\Omega;\mathbb{C}^3)$. As a consequence, classical Schauder estimates [**105**], [**103**] applied to (3.19) ((3.20) if $\omega = 0$) yield $\nabla q_E, \nabla q_H \in C^{0,\alpha}(\Omega;\mathbb{C}^3)$, which in turn imply that $(E, H) \in C^{0,\alpha}(\Omega;\mathbb{C}^3)^2$. The corresponding norm estimate follows from all the norm estimates related to the regularity results used in the argument. $\qquad\square$

Let us underline that the regularity assumptions on the coefficients given in the result above are optimal.

Remark 3.11. — Let $\Omega = B(0,1)$ be the unit ball and take $\alpha \in (0,1)$. Let

$$f \in L^\infty\big((-1,1);\mathbb{R}\big) \setminus C^\alpha\big((-1,1);\mathbb{R}\big)$$

such that $\Lambda^{-1} \le f \le \Lambda$ in $(-1,1)$. Let ε be defined by $\varepsilon(x) = f(x_1)$. Choosing $J = (-i\omega, 0, 0) \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$, observe that $E(x) = (f(x_1)^{-1}, 0, 0)$ and $H \equiv 0$ are weak solutions in $H(\operatorname{curl}, \Omega)^2$ to

$$\operatorname{curl} H = i\omega\varepsilon E + J \quad \text{in } \Omega, \quad \operatorname{curl} E = -i\omega H \quad \text{in } \Omega,$$

such that $E \notin C^{0,\alpha}(\Omega;\mathbb{C}^3)$. This shows that interior Hölder regularity cannot hold if ε is not Hölder continuous, even in the simplified case where ε depends only on one variable.

Let us now turn to local $W^{1,p}$ estimates for E and H.

Theorem 3.12. — *Take $p > 3$, $\Omega' \Subset \Omega$ and $\omega \in \mathbb{C}$. Let*

$$\mu \in W^{1,p}(\Omega;\mathbb{C}^{3\times 3}) \quad \text{and} \quad \varepsilon, \sigma \in W^{1,p}(\Omega;\mathbb{R}^{3\times 3})$$

satisfy (3.14) (and (3.15) if $\omega = 0$). Take $J, K \in W^{1,p}(\operatorname{div}, \Omega)$ (with $\operatorname{div} K = 0$ if $\omega = 0$). Let $(E, H) \in H(\operatorname{curl}, \Omega)^2$ be a weak solution of (3.13) (augmented

with (3.16) *if* $\omega = 0$). *Then* $E, H \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{C}^3)$ *and*

$$\|(E, H)\|_{W^{1,p}(\Omega')^2} \leq C\left(\|(E, H)\|_{L^2(\Omega)^2} + \|(J, K)\|_{W^{1,p}(\text{div}, \Omega)^2}\right)$$

for some $C > 0$ *depending only on* Ω, Ω', Λ, $|\omega|$ *and* $\|(\mu, \gamma)\|_{W^{1,p}(\Omega)^2}$.

Proof. — By the Sobolev embedding theorem and Theorem 3.10 we have $E, H \in L_{\text{loc}}^p(\Omega; \mathbb{C}^3)$. Thus, Lemma 3.1 applied to (3.21) yields

$$\Phi_E, \Phi_H \in W_{\text{loc}}^{2,p}(\Omega; \mathbb{C}^3).$$

Moreover, arguing as in the proof of Theorem 3.10 we obtain that $(\nabla q_E, \nabla q_H)$ belongs to $C(\Omega; \mathbb{C}^3)^2$. Differentiating (3.19) gives the elliptic equations

$$- \text{div}(\mu \nabla (\partial_i q_H)) = \text{div}((\partial_i \mu) \, \text{curl} \, \Phi_H + \mu \, \partial_i \, \text{curl} \, \Phi_H - i \omega^{-1} \partial_i K + \partial_i \mu \nabla q_H) \\ \text{in } \Omega,$$

$$- \text{div}(\gamma \nabla (\partial_i q_E)) = \text{div}((\partial_i \gamma) \, \text{curl} \, \Phi_E + \gamma \, \partial_i \, \text{curl} \, \Phi_E + i \, \partial_i J + \partial_i \gamma \nabla q_E) \ \text{in } \Omega.$$

If $\omega = 0$, (3.20) has to be considered instead, and the term $\text{div}(i \omega^{-1} \partial_i K)$ vanishes. Lemma 3.1 applied to these equations yields $\nabla q_E, \nabla q_H \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{C}^3)$, and we obtain as desired $E, H \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{C}^3)$. The corresponding norm estimates follow by applying all of the norm estimates related to the regularity results used in the argument. □

Remark 3.13. — Arguing as in Remark 3.11, we see that the regularity assumptions on the coefficients are minimal.

Finally, in the general case when ε, μ and σ are merely L^∞ let us show that E and H are in $L_{\text{loc}}^{2+\delta}(\Omega)$ for some $\delta > 0$.

Theorem 3.14 (Meyers' theorem for Maxwell's equations). — *Take* $\Omega' \Subset \Omega$ *and* $\omega \in \mathbb{C}$. *Let*

$$\mu \in L^\infty(\Omega; \mathbb{C}^{3 \times 3}) \quad \text{and} \quad \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$$

satisfy (3.14) (*and* (3.15) *if* $\omega = 0$). *Take* $J, K \in L^2(\Omega; \mathbb{C}^3)$ (*with* $\text{div} \, K = 0$ *if* $\omega = 0$). *There exist* $\delta > 0$ *depending only on* Ω *and* Λ *and* $C > 0$ *depending only on* Ω, Ω', Λ *and* $|\omega|$ *such that the following is true.*

Let $(E, H) \in H(\mathrm{curl}, \Omega)^2$ be a weak solution of (3.13) (augmented with (3.16) if $\omega = 0$). If $J \in L^{2+\delta}(\Omega; \mathbb{C}^3)$ then $E \in L^{2+\delta}_{\mathrm{loc}}(\Omega; \mathbb{C}^3)$ and

$$\|E\|_{L^{2+\delta}(\Omega')} \leq C\big(\|(E, H)\|_{L^2(\Omega)^2} + \|J\|_{L^{2+\delta}(\Omega)} + \|K\|_{L^2(\Omega)}\big);$$

and if $K \in L^{2+\delta}(\Omega; \mathbb{C}^3)$ then $H \in L^{2+\delta}_{\mathrm{loc}}(\Omega; \mathbb{C}^3)$ and

$$\|H\|_{L^{2+\delta}(\Omega')} \leq C\big(\|(E, H)\|_{L^2(\Omega)^2} + \|J\|_{L^2(\Omega)} + \|K\|_{L^{2+\delta}(\Omega)}\big).$$

Proof. — We follow the first steps of the proof of Theorem 3.9. We write

$$E = \nabla q_E + \mathrm{curl}\, \Phi_E \quad \text{and} \quad H = \nabla q_H + \mathrm{curl}\, \Phi_H$$

using the Helmholtz decomposition given in Corollary 3.8 . Let Ω'' be a smooth domain such that $\Omega' \Subset \Omega'' \Subset \Omega$. In view of (3.18) we have

$$\big\|(\mathrm{curl}\, \Phi_E, \mathrm{curl}\, \Phi_H)\big\|_{L^6(\Omega'')^2} \leq C\big(\|(E, H)\|_{L^2(\Omega)^2} + \|(J, K)\|_{L^2(\Omega)^2}\big),$$

for some $C > 0$ depending only on Ω, Ω', Λ and $|\omega|$. Thus, it remains to show that q_E and q_H are in $W^{1,2+\delta}(\Omega')$ for some $\delta > 0$. This is an immediate consequence of Theorem 3.5 applied to (3.19) ((3.20) if $\omega = 0$). $\qquad\square$

Remark 3.15. — The proofs of these regularity results highlight the very different roles played by the vector potentials Φ and the scalar potentials q in the Helmholtz decompositions

$$E = \nabla q_E + \mathrm{curl}\, \Phi_E, \quad H = \nabla q_H + \mathrm{curl}\, \Phi_H.$$

In all the cases previously considered, Φ_E or Φ_H are more regular than q_E and q_H. As far as the $C^{0,\alpha}$ ($0 < \alpha < \frac{1}{2}$) estimates are concerned, we obtain directly that Φ_E and Φ_H are in fact in $C^{1,1/2}$, which in turn implies that $\mathrm{curl}\, \Phi_E$ and $\mathrm{curl}\, \Phi_H$ are in $C^{0,\frac{1}{2}}$. Regarding the $W^{1,p}$ result, we obtain that the vector potentials are in fact in $C^{2,\alpha}$, so that $\mathrm{curl}\, \Phi_E$ and $\mathrm{curl}\, \Phi_H$ are in $C^{1,\alpha}$, a much smaller space than $W^{1,p}$. In the Meyers' theorem for Maxwell's equations, Sobolev embeddings show that $\mathrm{curl}\, \Phi_E, \mathrm{curl}\, \Phi_H$ are in L^6, which is a much higher integrability than the one of ∇q_E and ∇q_H in general.

The regularity results we established depend essentially on proving regularity results for the scalar potentials, namely ∇q_E and ∇q_H. We did so by using the elliptic equations they satisfy. As a consequence, as mentioned in Section 3.1,

the crucial aspects of the study of the general case corresponding to a non-zero frequency $\omega \in \mathbb{C}$ are substantially equivalent to those in the case of the conductivity equation, corresponding to the case $\omega = 0$.

As we shall see in Chapter 4, the same phenomenon occurs when studying asymptotic estimates of the solutions due to small inclusions in the parameters: the leading order effect will be expressed in terms of scalar potentials, and the vector potentials will affect only the higher order terms.

3.4. Well-posedness for Maxwell's equations

For completeness and future reference in Chapter 4, we recall classical well-posedness results for Maxwell's system of equations. The reader is referred to [**211**], [**138**], [**198**], [**163**], [**4**] for full details.

Consider problem (3.1) augmented with Dirichlet boundary conditions

$$(3.22) \qquad \begin{cases} \operatorname{curl} E = i\omega\mu H + K & \text{in } \Omega, \\ \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E + J & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases}$$

where $J, K \in L^2(\Omega; \mathbb{C}^3)$, $\varphi \in H(\operatorname{curl}, \Omega)$ and $\mu, \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3\times3})$ and satisfy

$$(3.23) \qquad \begin{cases} \Lambda^{-1}|\xi|^2 \le \xi \cdot \mu\xi, \quad \Lambda^{-1}|\xi|^2 \le \xi \cdot \varepsilon\xi, \quad \xi \in \mathbb{R}^3, \\ \|(\sigma, \varepsilon, \mu)\|_{L^\infty(\Omega; \mathbb{R}^{3\times3})^3} \le \Lambda, \quad \mu = \mu^T, \ \varepsilon = \varepsilon^T, \ \sigma = \sigma^T \end{cases}$$

for some $\Lambda > 0$ and either

$$(3.24) \qquad\qquad\qquad \sigma = 0 \quad \text{in } \Omega,$$

or

$$(3.25) \qquad\qquad\qquad \Lambda^{-1}|\xi|^2 \le \xi \cdot \sigma\xi, \quad \xi \in \mathbb{R}^3.$$

Remark 3.16. — By the Fredholm theory, if the problem is not well-posed, then there exists $E, H \in H(\operatorname{curl}, \Omega)$ such that

$$\begin{cases} \operatorname{curl} E = i\omega\mu H & \text{in } \Omega, \\ \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E & \text{in } \Omega, \\ E \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

see [**211**], [**198**], [**4**].

The main well-posedness result in the non conductive case reads as follows.

Proposition 3.17. — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded and $C^{1,1}$ domain and μ, ε, σ in $L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ be such that (3.23) and (3.24) hold true. There exists a discrete set of eigenvalues $\Sigma \subseteq \mathbb{R}_+$ such that if $\omega \in \mathbb{R}_+ \setminus \Sigma$ then for any $J, K \in L^2(\Omega; \mathbb{C}^3)$ and $\varphi \in H(\mathrm{curl}, \Omega)$ problem (3.22) admits a unique solution $(E, H) \in H(\mathrm{curl}, \Omega)^2$ and*

$$\|(E, H)\|_{H(\mathrm{curl}, \Omega)^2} \leq C \big(\|\varphi\|_{H(\mathrm{curl}, \Omega)} + \|(J, K)\|_{L^2(\Omega)^2} \big)$$

for some $C > 0$ depending only on Ω, ω and Λ.

The main well-posedness result in the dissipative case reads as follows.

Proposition 3.18. — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded and $C^{1,1}$ domain, $\omega > 0$ and μ, ε, σ in $L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ be such that (3.23) and (3.25) hold true. Then for any J, K in $L^2(\Omega; \mathbb{C}^3)$ and $\varphi \in H(\mathrm{curl}, \Omega)$ problem (3.22) admits a unique solution (E, H) in $H(\mathrm{curl}, \Omega)^2$ and*

$$\|(E, H)\|_{H(\mathrm{curl}, \Omega)^2} \leq C \big(\|\varphi\|_{H(\mathrm{curl}, \Omega)} + \|(J, K)\|_{L^2(\Omega)^2} \big)$$

for some $C > 0$ depending only on Ω, ω and Λ.

The case $\omega = 0$ is somehow peculiar since additional assumptions are required on the sources: it is considered in the following result.

Proposition 3.19. — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$. Let $\mu, \varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ be such that (3.23) and (3.25) hold true. Take $J, K \in L^2(\Omega; \mathbb{C}^3)$ and $\varphi \in H(\mathrm{curl}, \Omega)$ with $\mathrm{div}\, K = 0$ in Ω and $K \cdot \nu = \mathrm{curl}\, \varphi \cdot \nu$ on $\partial\Omega$. Then, the problem*

$$(3.26) \quad \begin{cases} \mathrm{curl}\, E = K & \text{in } \Omega, \\ \mathrm{curl}\, H = \sigma E + J & \text{in } \Omega, \\ \mathrm{div}(\mu H) = 0 & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \\ (\mu H) \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $(E, H) \in H(\mathrm{curl}, \Omega)^2$ and

$$\|(E, H)\|_{H(\mathrm{curl}, \Omega)^2} \leq C \big(\|\varphi\|_{H(\mathrm{curl}, \Omega)} + \|(J, K)\|_{L^2(\Omega)^2} \big)$$

for some $C > 0$ depending only on Ω and Λ.

Proof. — In what follows various positive constants depending only on Ω and Λ will be denoted by C. Write

$$\widetilde{E} = E - \varphi \quad \text{and} \quad \widetilde{K} = K - \operatorname{curl} \varphi.$$

By Lemma 3.7, it is enough to look for solutions $E, H \in H(\operatorname{curl}, \Omega)$ of the form

$$\widetilde{E} = \nabla q_E + \operatorname{curl} \Phi_E, \quad H = \nabla q_H + \operatorname{curl} \Phi_H,$$

with $q_E \in H_0^1(\Omega; \mathbb{C})$, $q_H \in H^1(\Omega; \mathbb{C})$, $\Phi_E, \Phi_H \in H^1(\Omega; \mathbb{C}^3)$ and $\Phi_H \times \nu = 0$ on $\partial \Omega$. Set $\Psi_E = \operatorname{curl} \Phi_E$, assuming momentarily the existence of such a vector potential. Since q_E is constant on $\partial \Omega$ we have

$$\Psi_E \times \nu = \widetilde{E} \times \nu - \nabla q_E \times \nu = 0 \quad \text{on } \partial \Omega.$$

Moreover, $\operatorname{div} \Psi_E = \operatorname{div} \operatorname{curl} \Phi_E = 0$ in Ω and $\operatorname{curl} \Psi_E = \operatorname{curl} \widetilde{E} = \widetilde{K}$ from the first equation of (3.26). Thus Lemma 3.6 shows that $\operatorname{curl} \Phi_E \in H^1(\Omega; \mathbb{C}^3)$ and

$$(3.27) \qquad \| \operatorname{curl} \Phi_E \|_{H^1(\Omega)} \leq C \big(\|K\|_{L^2(\Omega)} + \| \operatorname{curl} \varphi \|_{L^2(\Omega)} \big).$$

The existence of Ψ_E follows from the fact that $\operatorname{div} \widetilde{K} = 0$ in Ω and $\widetilde{K} \cdot \nu = 0$ on $\partial \Omega$ [**106**, Chapter 1, Theorem 3.6]. Moreover, by [**106**, Chapter 1, Theorem 3.5], one has $\Psi_E = \operatorname{curl} \Phi_E$ for some $\Phi_E \in H^1(\Omega; \mathbb{C}^3)$ such that $\operatorname{div} \Phi_E = 0$ in Ω and $\Phi_E \cdot \nu = 0$ on $\partial \Omega$. Thus, $\operatorname{curl} \Phi_E$ is now uniquely determined, and the first and fourth equations of (3.26) are automatically satisfied.

The second equation of (3.26) implies that q_E satisfies

$$(3.28) \qquad \begin{cases} -\operatorname{div}(\sigma \nabla q_E) = \operatorname{div}(\sigma \operatorname{curl} \Phi_E + \sigma \varphi + J) & \text{in } \Omega, \\ q_E = 0 & \text{on } \partial \Omega. \end{cases}$$

The Lax–Milgram theorem provides existence and uniqueness of $q_E \in H_0^1(\Omega)$, with the norm estimate

$$\|q_E\|_{H^1(\Omega)} \leq C(\| \operatorname{curl} \Phi_E \|_{L^2(\Omega; \mathbb{C}^3)} + \|(J, \varphi)\|_{L^2(\Omega)^2}).$$

As a consequence, by using the estimate on $\operatorname{curl} \Phi_E$ obtained above we have

$$\|q_E\|_{H^1(\Omega)} \leq C \big(\|\varphi\|_{H(\operatorname{curl}, \Omega)} + \|(K, J)\|_{L^2(\Omega)^2} \big),$$

$$\|E\|_{H(\operatorname{curl}, \Omega)} \leq C \big(\|\varphi\|_{H(\operatorname{curl}, \Omega)} + \|(K, J)\|_{L^2(\Omega)^2} \big).$$

Inserting $H = \nabla q_H + \Psi_H$ with $\Psi_H = \operatorname{curl} \Phi_H$ into the second equation of (3.26) we have

$$\operatorname{div} \Psi_H = 0 \quad \text{in } \Omega, \quad \operatorname{curl} \Psi_H = \sigma E + J \quad \text{in } \Omega.$$

Moreover, since $\Phi_H \times \nu = 0$ on $\partial\Omega$, by [**163**, equation (3.52)] we have

$$\Psi_H \cdot \nu = (\operatorname{curl} \Phi_H) \cdot \nu = \operatorname*{div}_{\partial\Omega}(\Phi_H \times \nu) = 0 \quad \text{on } \partial\Omega.$$

We apply Lemma 3.6 and deduce that $\operatorname{curl} \Phi_H \in H^1(\Omega; \mathbb{C}^3)$ and

$$\| \operatorname{curl} \Phi_H \|_{H^1(\Omega)} \le C \|\sigma E + J\|_{L^2(\Omega)} \le C \big(\|\varphi\|_{H(\operatorname{curl},\Omega)} + \|(K,J)\|_{L^2(\Omega)^2} \big).$$

The existence of Ψ_H follows from [**106**, Chap. 1, Theorem 3.5], since $\operatorname{div}(\sigma E + J) = 0$ in Ω. Moreover, by [**106**, Chapter 1, Theorem 3.6], $\Psi_H = \operatorname{curl} \Phi_H$ for some Φ_H in $H^1(\Omega; \mathbb{C}^3)$ such that $\operatorname{div} \Phi_H = 0$ in Ω and $\Phi_H \times \nu = 0$ on $\partial\Omega$. The second equation of (3.26) is now automatically satisfied, and $\operatorname{curl} \Phi_H$ is uniquely determined.

The third and fifth equations of (3.26) now imply that q_H satisfies

$$\begin{cases} -\operatorname{div}(\mu \nabla q_H) = \operatorname{div}(\mu \operatorname{curl} \Phi_H) & \text{in } \Omega, \\ -\mu \nabla q_H \cdot \nu = \mu \operatorname{curl} \Phi_H \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Thus, standard elliptic theory immediately yields existence and uniqueness for this problem in $H^1(\Omega; \mathbb{C})/\mathbb{C}$ with the norm estimate

$$\|\nabla q_H\|_{L^2(\Omega)} \le C \| \operatorname{curl} \Phi_H \|_{L^2(\Omega)}.$$

As a consequence, by using the estimate on $\operatorname{curl} \Phi_H$ obtained above we have

$$\|\nabla q_H\|_{L^2(\Omega)} \le C \big(\|\varphi\|_{H(\operatorname{curl},\Omega)} + \|(K,J)\|_{L^2(\Omega)^2} \big),$$

$$\|H\|_{H(\operatorname{curl},\Omega)} \le C \big(\|\varphi\|_{H(\operatorname{curl},\Omega)} + \|(K,J)\|_{L^2(\Omega)^2} \big).$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.20. — Consider the particular case when $K = 0$ and $\varphi = \nabla v$ for some $v \in H^1(\Omega; \mathbb{C})$. By (3.27) we can write $E = \nabla u$ where $u = q_E + v \in H^1(\Omega; \mathbb{C})$, since $\operatorname{curl} \Phi_E = 0$. Moreover, by (3.28) the electric potential u is the unique solution to

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = \operatorname{div} J & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega. \end{cases}$$

In other words, as already mentioned above, the case $\omega = 0$ in Maxwell's system corresponds to the conductivity equation.

CHAPTER 4

PERTURBATIONS OF SMALL VOLUME FOR MAXWELL'S EQUATIONS IN A BOUNDED DOMAIN

4.1. Introduction

Consider time-harmonic electromagnetic fields E and H travelling in a medium Ω with permittivity $\varepsilon \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ and permeability $\mu \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$. Both ε and μ are symmetric and positive definite matrix valued functions. According to Maxwell's system of equations (see (3.1) with $\sigma = 0$), they verify

$$\operatorname{curl} E = i\omega\mu H, \quad \operatorname{curl} H = -i\omega\varepsilon E \quad \text{in } \Omega.$$

Suppose that at frequency ω, when a boundary condition is imposed on one of the fields, say

$$E \times \nu = \varphi \times \nu \quad \text{on } \partial\Omega,$$

the problem is well posed. If the medium is perturbed – by a focused pressure wave for example – or if the coefficients present defects within the domain, in a small set D such that $D \Subset \Omega$ and $|D| \ll |\Omega|$, the physical parameters can be written

$$\mu_D = \widetilde{\mu}\,\mathbb{1}_D + (1 - \mathbb{1}_D)\mu, \quad \varepsilon_D = \widetilde{\varepsilon}\,\mathbb{1}_D + (1 - \mathbb{1}_D)\varepsilon,$$

where $\widetilde{\mu}$ and $\widetilde{\varepsilon}$ are the permeability and the permittivity within the inclusion D, respectively. The electromagnetic fields then become E_D and H_D, satisfying

$$\operatorname{curl} E_D = i\omega\mu_D H_D, \quad \operatorname{curl} H_D = -i\omega\varepsilon_D E_D \text{ in } \Omega, \quad E_D \times \nu = \varphi \times \nu \text{ on } \partial\Omega.$$

The effect of these defects can be measured on the boundary of the domain. Indeed, writing

$$B_D = \int_{\partial\Omega} (E_D \times \nu) \cdot H_D \, \mathrm{d}\sigma \quad \text{and} \quad B = \int_{\partial\Omega} (E \times \nu) \cdot H \, \mathrm{d}\sigma,$$

an integration by parts shows that

(4.1) $$B_D - B = \mathrm{i}\omega \int_D (\widetilde{\mu} - \mu)H \cdot H_D - (\widetilde{\varepsilon} - \varepsilon)E \cdot E_D \, \mathrm{d}x.$$

In other words, from external boundary difference measurements, localised information on the defects is available. A heuristic approximation, the so-called Born approximation, is then to consider that $E_D \approx E$ and $H_D \approx H$ and to linearise the dependence on D by writing

$$B_D - B \approx \mathrm{i}\omega \int_D (\widetilde{\mu} - \mu)H \cdot H - (\widetilde{\varepsilon} - \varepsilon)E \cdot E \, \mathrm{d}x.$$

In particular, if the defect is localised near $x_0 \in \Omega$ this yields

$$\frac{B_D - B}{|D|} \approx \mathrm{i}\omega\Big(\big(\widetilde{\mu}(x_0) - \mu(x_0)\big)H(x_0) \cdot H(x_0) - \big(\widetilde{\varepsilon}(x_0) - \varepsilon(x_0)\big)E(x_0) \cdot E(x_0)\Big),$$

leading (not unlike what we saw in Chapter 1) to an internal density information. It turns out that this heuristic argument is not correct except when $\widetilde{\varepsilon} - \varepsilon$ and $\widetilde{\mu} - \mu$ are also small, leading to a small amplitude and small volume fraction approximation. The correct first order expansion, without assumptions on the smallness of the amplitude of the defects, involves polarisation tensors. The derivation of this approximation is the subject of this chapter.

Using the regularity results obtained in Chapter 3, we derive the leading order term in $|D|$ of the asymptotic expansion of $\mathbb{1}_D(E_D - E)$ and $\mathbb{1}_D(H_D - H)$, for general internal inclusions bounded in L^∞ on a measurable set D located within Ω, when the background medium parameters ε and μ are sufficiently smooth (namely, $C^{0,\alpha}$ or $W^{1,p}$ with $p > 3$). Several expansions of this type are available in the literature, for conductivity, elasticity, cavities, and electromagnetic fields [96], [164], [165], [79], [23], [31, 108]. The existing results for Maxwell's system are somewhat less general [23], [108], and use a slightly different approach – which typically requires a constant background medium.

The method used here was first introduced in [**73**] for the conductivity problem, which corresponds to the case $\omega = 0$. An additional ingredient is the Helmholtz decomposition of the fields. We prove that the essential features of the problem are captured by the elliptic equations satisfied by the scalar potentials, for which the method of [**73**] can be applied.

4.2. Model, assumptions, and preliminary results

Let us now consider the problem in full generality, using the notation of Section 3.1. Suppose Ω is a simply connected and connected domain with a connected boundary $\partial\Omega$ of class $C^{1,1}$, and let $E, H \in H(\mathrm{curl}, \Omega)$ be the solutions of

$$(4.2) \qquad \begin{cases} \mathrm{curl}\, E = i\omega\mu H + K & \text{in } \Omega, \\ \mathrm{curl}\, H = -i\gamma E + J & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases}$$

where $\omega \in \mathbb{C}$, $K, J \in L^2(\Omega; \mathbb{C}^3)$, $\varphi \in H(\mathrm{curl}, \Omega)$, $\gamma = \omega\varepsilon + i\sigma$ and $\mu \in L^\infty(\Omega; \mathbb{C}^{3\times 3})$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ are symmetric tensors such that for all $\zeta \in \mathbb{R}^3$ there holds

$$(4.3) \qquad \begin{cases} \Lambda^{-1}|\zeta|^2 \le \zeta \cdot (\mathrm{Re}\,\mu)\zeta, \quad \Lambda^{-1}|\zeta|^2 \le \zeta \cdot \varepsilon\zeta, \\ |\mu| + |\varepsilon| + |\sigma| \le \Lambda \quad \text{a.e. in } \Omega, \\ \mu = \mu^T, \quad \varepsilon = \varepsilon^T, \quad \sigma = \sigma^T \quad \text{a.e. in } \Omega \end{cases}$$

for some $\Lambda > 0$.

If $\omega = 0$, we assume additionally that for every $\zeta \in \mathbb{R}^3$

$$(4.4) \qquad\qquad \Lambda^{-1}|\zeta|^2 \le \zeta \cdot \sigma\zeta \quad \text{a.e. in } \Omega,$$

$\mathrm{div}\, K = 0$ in Ω and $K \cdot \nu = \mathrm{curl}\, \varphi \cdot \nu$ on $\partial\Omega$ and we also impose the following two equations

$$(4.5) \qquad \begin{cases} \mathrm{div}(\mu H) = 0 & \text{in } \Omega, \\ (\mu H) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

We suppose that the map $(J, K, \varphi) \mapsto (E, H)$ is well defined and continuous, namely that (4.2) is well posed. More precisely, we assume that for any K, J

in $L^2(\Omega; \mathbb{C}^3)$ and $\varphi \in H(\mathrm{curl}, \Omega)$ (such that $\mathrm{div}\, K = 0$ in Ω and $K \cdot \nu = \mathrm{curl}\, \varphi \cdot \nu$ on $\partial\Omega$ if $\omega = 0$), the solution $(E, H) \in H(\mathrm{curl}, \Omega)^2$ satisfies

$$(4.6) \qquad \|(E, H)\|_{H(\mathrm{curl},\Omega)^2} \leq C_0 \big(\|(K, J)\|_{L^2(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)} \big)$$

for some $C_0 > 0$. Sufficient conditions for this problem to be well posed are given in Section 3.4. Typically, well-posedness holds for ω outside a discrete set of values (not containing 0) or if μ is real and σ is strictly positive.

Consider now that defects are present within the medium, namely the electromagnetic fields satisfy

$$(4.7) \qquad \begin{cases} \mathrm{curl}\, E_D = i\omega\mu_D H_D + K & \text{in } \Omega, \\ \mathrm{curl}\, H_D = -i\gamma_D E_D + J & \text{in } \Omega, \\ E_D \times \nu = \varphi \times \nu & \text{on } \partial\Omega, \end{cases}$$

(augmented with

$$(4.8) \qquad \begin{cases} \mathrm{div}(\mu_D H_D) = 0 & \text{in } \Omega, \\ (\mu_D H_D) \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

if $\omega = 0$), where

$$\mu_D = \mu(1 - \mathbb{1}_D) + \widetilde{\mu}\mathbb{1}_D, \quad \gamma_D = \gamma(1 - \mathbb{1}_D) + \widetilde{\gamma}\mathbb{1}_D \quad \text{in } \Omega,$$

where $\mathbb{1}_D$ is the characteristic function of a measurable set D located within Ω, $\widetilde{\gamma} = \omega\widetilde{\varepsilon} + i\widetilde{\sigma}$, $\widetilde{\mu} \in L^\infty(\Omega; \mathbb{C}^{3\times3})$ and $\widetilde{\varepsilon}, \widetilde{\sigma} \in L^\infty(\Omega; \mathbb{R}^{3\times3})$ are symmetric and satisfy (4.3) (and (4.4) if $\omega = 0$). We suppose that the inclusion D is not close to the boundary, namely $D \subseteq \Omega_0$ for some smooth connected and simply connected subdomain Ω_0 of Ω such that $\Omega_0 \Subset \Omega$.

Let us show that this perturbed problem is also well-posed, provided that $|D|$ is sufficiently small.

Lemma 4.1. — *Under the above assumptions, there exists $d_0 > 0$ depending only on Ω, Ω_0, Λ, C_0 and $|\omega|$ such that when*

$$(4.9) \qquad\qquad\qquad |D| \leq d_0$$

problem (4.7) (*augmented with* (4.8) *if* $\omega = 0$) *admits a unique solution, which satisfies*

$$\|(E_D, H_D)\|_{H(\mathrm{curl},\Omega)^2} \leq C\big(\|(K, J)\|_{L^2(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)}\big)$$

for some $C > 0$ *depending only on* Ω, Ω_0, Λ, C_0 *and* $|\omega|$. *Furthermore, there exist* $\delta > 0$ *depending only on* Ω *and* Λ, *and* $C > 0$ *depending only on* Ω, Ω_0, Λ, C_0 *and* $|\omega|$ *such that when* $J, K \in L^{2+\delta}(\Omega; \mathbb{C}^3)$ *then*

$$\|(E_D, H_D) - (E, H)\|_{H(\mathrm{curl},\Omega)^2} \leq C|D|^{\frac{\delta}{4+2\delta}}\big(\|(K, J)\|_{L^{2+\delta}(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)}\big),$$

where (E, H) *is the solution of* (4.2) (*augmented with* (4.5) *if* $\omega = 0$).

Proof. — Note that when Proposition 3.17 applies, we know that the number of resonant frequencies ω for which (4.7) is not well-posed is a discrete set. The issue at hand regarding well-posedness is thus the behaviour of the resonances when $|D| \to 0$. Furthermore, by Remark 3.16, for a given ω it is sufficient to establish uniqueness when $J = K = \varphi = 0$ to establish well-posedness for any other J, K and φ.

With an abuse of notation, several positive constants depending only on Ω, Ω_0, Λ, C_0 and $|\omega|$ will be denoted by C. Assume that $(E_D, H_D) \in H(\mathrm{curl}, \Omega)^2$ is a solution of (4.7) (augmented with (4.8) if $\omega = 0$). According to Theorem 3.14, there exists $\delta > 0$ depending only on Ω and Λ such that when $J, K \in L^{2+\delta}(\Omega; \mathbb{C}^3)$ we have

$$(4.10) \quad \|(E_D, H_D)\|_{L^{2+\delta}(\Omega_0)^2}$$
$$\leq C\big(\|(E_D, H_D)\|_{L^2(\Omega)^2} + \|(K, J)\|_{L^{2+\delta}(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)}\big).$$

Set $W_D = E_D - E$ and $Q_D = H_D - H$. The pair $(W_D, Q_D) \in H(\mathrm{curl}, \Omega)^2$ is a solution of

$$\begin{cases} \mathrm{curl}\, W_D = i\omega\mu Q_D + i\omega\, \mathbb{1}_D(\widetilde{\mu} - \mu)H_D & \text{in } \Omega, \\ \mathrm{curl}\, Q_D = -i\gamma W_D - i\mathbb{1}_D(\widetilde{\gamma} - \gamma)E_D & \text{in } \Omega, \\ W_D \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, using well-posedness for problem (4.2), namely estimate (4.6), we find

$$\|(W_D, Q_D)\|_{H(\mathrm{curl},\Omega)^2} \leq C\big(\|\omega\, \mathbb{1}_D(\widetilde{\mu} - \mu)H_D\|_{L^2(\Omega)} + \|\mathbb{1}_D(\widetilde{\gamma} - \gamma)E_D\|_{L^2(\Omega)}\big)$$
$$\leq C\|(\mathbb{1}_D H_D, \mathbb{1}_D E_D)\|_{L^2(\Omega_0)^2}.$$

Hence, by Hölder's inequality and (4.10), we obtain

$$(4.11) \quad \|(W_D, Q_D)\|_{H(\mathrm{curl},\Omega)^2} \leq C|D|^{\frac{\delta}{4+2\delta}} \cdot \|(E_D, H_D)\|_{L^{2+\delta}(\Omega_0)^2}$$

$$\leq C|D|^{\frac{\delta}{4+2\delta}} \Big(\|(E_D, H_D)\|_{L^2(\Omega)^2} + \|(K, J)\|_{L^{2+\delta}(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)} \Big).$$

Consider now the case when $J = K = \varphi = 0$. Then (4.11) is simply

$$\|(E_D, H_D)\|_{H(\mathrm{curl},\Omega)^2} \leq C|D|^{\frac{\delta}{4+2\delta}} \cdot \|(E_D, H_D)\|_{H(\mathrm{curl},\Omega)^2}.$$

Thus, when

$$(4.12) \qquad\qquad |D| \leq (C+1)^{-(4+2\delta)/\delta},$$

we find $\|(E_D, H_D)\|_{H(\mathrm{curl},\Omega)^2} = 0$. As a consequence, problem (4.7) is also well posed when (4.12) holds. In particular,

$$\|(E_D, H_D)\|_{H(\mathrm{curl},\Omega)^2} \leq C \big(\|(K, J)\|_{L^2(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)} \big).$$

Moreover, (4.11) implies

$$\|(E_D, H_D) - (E, H)\|_{H(\mathrm{curl},\Omega)^2} \leq C|D|^{\frac{\delta}{4+2\delta}} \big(\|(K, J)\|_{L^{2+\delta}(\Omega)^2} + \|\varphi\|_{H(\mathrm{curl},\Omega)} \big),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This preliminary result confirms that this is a perturbation problem, since E_D and H_D converge strongly to E and H in $H(\mathrm{curl}, \Omega)$, with at least the rate dictated by the built-in higher integrability of Maxwell's system. Without additional structural information on μ and γ, the exact rate of convergence may vary [**59**], leaving little hope for a general result for the first order term. However, much more can be said when μ and γ are known to be smoother, without further assumptions on $\widetilde{\mu}$ and $\widetilde{\gamma}$.

4.3. Main results

Let $p_D^E \in H^1(\Omega; \mathbb{C})$ be solution of

$$(4.13) \qquad \begin{cases} -\operatorname{div}(\gamma_D \nabla p_D^E) = \operatorname{div}(\mathbb{1}_D (\widetilde{\gamma} - \gamma) E) & \text{in } \Omega, \\ p_D^E = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $p_D^H \in H^1(\Omega;\mathbb{C})$ be solution of

$$(4.14) \qquad \begin{cases} -\operatorname{div}(\mu_D \nabla p_D^H) = \operatorname{div}\left(\mathbb{1}_D(\widetilde{\mu} - \mu)H\right) & \text{in } \Omega, \\ \mu_D \nabla p_D^H \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The potential p_E^D is uniquely determined, while p_H^D is determined up to a multiplicative constant.

Theorem 4.2. — *Assume that* (4.3), (4.6) *and* (4.9) *(and* (4.4) *if* $\omega = 0$*) hold, and that additionally*

$$(4.15) \qquad \mu, \gamma \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^{3\times 3}) \quad \text{and} \quad K, J \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$$

for some $\alpha \in (0, \frac{1}{2}]$ *(with* $\operatorname{div} K = 0$ *in* Ω *if* $\omega = 0$*). Take* $\varphi \in H(\operatorname{curl}, \Omega)$ *(such that* $K \cdot \nu = \operatorname{curl}\varphi \cdot \nu$ *on* $\partial\Omega$ *if* $\omega = 0$*), and define*

$$M_\alpha = \|\varphi\|_{H(\operatorname{curl},\Omega)} + \|(J,K)\|_{C^{0,\alpha}(\overline{\Omega})^2}.$$

The solution $(E_D, H_D) \in H(\operatorname{curl}, \Omega)^2$ *to* (4.7) *(augmented with* (4.8) *if* $\omega = 0$*) admits the following expansion*

$$E_D = E + \nabla p_D^E + R_D^E \quad \text{in } \Omega,$$
$$H_D = H + \nabla p_D^H + R_D^H \quad \text{in } \Omega,$$

where the remainder terms R_E *and* R_H *are bounded by*

$$\|R_D^E\|_{L^2(\Omega)} + \|R_D^H\|_{L^2(\Omega)} \le C|D|^{\frac{1}{2}+\delta} \cdot \left\|\max\left(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma|\right)\right\|_{L^\infty(D)} \cdot M_\alpha,$$

for some $\delta > 0$ *depending only on* Ω *and* Λ *and some* $C > 0$ *depending only on* Ω, Ω_0, Λ, C_0, $\|(\mu,\gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$ *and* $|\omega|$.

Remark 4.3. — Assumption (4.15) may be relaxed. It is sufficient to assume the regularity of the parameters and of the sources in Ω_1, for some smooth domain Ω_1 such that $\Omega_0 \Subset \Omega_1 \subseteq \Omega$.

Any other choice of boundary conditions for p_D^E and P_D^H would lead to similar results; the above boundary conditions were chosen in order to make the analysis as simple as possible. Note that the magnetic potential p_D^H is always independent of the frequency; this is true of the electric potential p_D^E only when $\sigma = \widetilde{\sigma} \equiv 0$.

Under slightly stronger assumptions on the regularity of the coefficients and of the source terms, we can make this decomposition even more explicit. Let

$$P_D^\gamma = [\nabla p_\gamma^1, \nabla p_\gamma^2, \nabla p_\gamma^3] \in L^2(\Omega; \mathbb{C}^{3\times 3}), \quad P_D^\mu = [\nabla p_\mu^1, \nabla p_\mu^2, \nabla p_\mu^3] \in L^2(\Omega; \mathbb{C}^{3\times 3})$$

be the matrix valued maps defined as follows. For $k = 1, 2, 3$, let $p_\gamma^k \in H^1(\Omega; \mathbb{C})$ be solution of

$$(4.16) \qquad \begin{cases} -\operatorname{div}(\gamma_D \nabla p_\gamma^k) = \operatorname{div}(\mathbb{1}_D(\gamma_D - \gamma)\mathbf{e}_k) & \text{in } \Omega, \\ p_\gamma^k = 0 & \text{on } \partial\Omega, \end{cases}$$

and for $k = 1, 2, 3$, let $p_\mu^k \in H^1(\Omega; \mathbb{C})$ be solution of

$$(4.17) \qquad \begin{cases} -\operatorname{div}(\mu_D \nabla p_\mu^k) = \operatorname{div}(\mathbb{1}_D(\mu_D - \mu)\mathbf{e}_k) & \text{in } \Omega, \\ \mu_D \nabla p_\mu^k \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = I_3$ is the 3×3 identity matrix.

Theorem 4.4. — *Assume that* (4.3), (4.6) *and* (4.9) (*and* (4.4) *if* $\omega = 0$) *hold, and that additionally*

$$(4.18) \qquad \mu, \gamma \in W^{1,p}(\Omega; \mathbb{C}^{3\times 3}) \quad \text{and} \quad K, J \in W^{1,p}(\operatorname{div}, \Omega)$$

for some $p > 3$ (*with* $\operatorname{div} K = 0$ *in* Ω *if* $\omega = 0$). *Take* $\varphi \in H(\operatorname{curl}, \Omega)$ (*such that* $K \cdot \nu = \operatorname{curl}\varphi \cdot \nu$ *on* $\partial\Omega$ *if* $\omega = 0$), *and set*

$$M_p = \|\varphi\|_{H(\operatorname{curl}, \Omega)} + \|(J, K)\|_{W^{1,p}(\operatorname{div}, \Omega)^2}.$$

The solution $(E_D, H_D) \in H(\operatorname{curl}, \Omega)^2$ *to* (4.7) (*augmented with* (4.8) *if* $\omega = 0$) *admits the following expansion*

$$E_D = (I_3 + P_D^\gamma)E + \widetilde{R}_D^E \quad \text{in } \Omega,$$
$$H_D = (I_3 + P_D^\mu)H + \widetilde{R}_D^E \quad \text{in } \Omega,$$

where the remainder terms $\widetilde{R}_E$ *and* $\widetilde{R}_H$ *are bounded by*

$$\|(\widetilde{R}_D^E, \widetilde{R}_D^H)\|_{L^2(\Omega_0)^2} \leq C|D|^{\frac{1}{2}+\delta}\|\max(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma|)\|_{L^\infty(D)} M_p,$$

for some $\delta > 0$ *depending only on* Ω, Λ *and* p *and some* $C > 0$ *depending only on* Ω, Ω_0, Λ, C_0, $\|(\mu, \gamma)\|_{W^{1,p}(\Omega)^3}$ *and* $|\omega|$.

Remark 4.5. — The matrix valued functions $I_3 + P_D^\gamma$ and $I_3 + P_D^\mu$ (once rescaled by $|D|$, and passing to the limit as $|D| \to 0$) are called polarisation tensors. They are independent of the electric field imposed, and depend only on D, $\widetilde{\gamma}$, γ, $\widetilde{\mu}$ and μ. The above boundary conditions on $\partial\Omega$ were chosen for simplicity, but in the scaled limit the dependence on Ω and on the boundary values disappears.

Their analytic expression can be derived for several basic geometries, see *e.g.* [**158**], [**67**], [**31**], [**204**]. For example, when D is a ball, and $\gamma, \widetilde{\gamma} \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ (namely, they are scalar, real valued and Hölder continuous), then

$$I_3 + P_D^\gamma = \frac{3\gamma}{2\gamma + \gamma_D} I_3 \big(1 + O(|D|^\beta)\big) \quad \text{in} \ \ \Omega$$

for some $\beta > 0$. This is the Claussius–Mossotti, or the Maxwell–Garnett formula, see *e.g.* [**79**], [**158, 73**].

For general shapes, P_D^γ (and *mutatis mutandis* P_D^μ) satisfies *a priori* bounds, known as Hashin–Shtrikman bounds in the theory of composites (see [**166**], [**158**]). In particular, when $\gamma, \widetilde{\gamma} \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ there holds

$$\text{tr}\big(P_D^\gamma(x)\big) \leq |D|\Big(d - 1 + \frac{\gamma(x)}{\widetilde{\gamma}(x)}\Big)\big(1 + o(1)\big) \quad \text{in} \ \ D,$$

$$\text{tr}\big((P_D^\gamma(x))^{-1}\big) \leq |D|\Big(d - 1 + \frac{\widetilde{\gamma}(x)}{\gamma(x)}\Big)\big(1 + o(1)\big) \quad \text{in} \ \ D,$$

see [**147**], [**74**].

Thus $P_D^\gamma E$ is not a term that can be neglected as it always contributes to the leading order term. We can write the following expansions, which clarify the difference between this approach and the Born approximation.

Corollary 4.6. — *Assume that the hypotheses of Theorem 4.4 hold true. For every* Φ *in* $L^\infty(\Omega_0; \mathbb{C}^3)$ *we have*

$$\int_D (\widetilde{\gamma} - \gamma) E_D \cdot \Phi \, dx = \int_D (\widetilde{\gamma} - \gamma)(I_3 + P_D^\gamma) E \cdot \Phi \, dx + O\big(|D|^{1+\delta} \|\Phi\|_{L^\infty(D)}\big),$$

$$\int_D (\widetilde{\mu} - \mu) H_D \cdot \Phi \, dx = \int_D (\widetilde{\mu} - \mu)(I_3 + P_D^\mu) H \cdot \Phi \, dx + O\big(|D|^{1+\delta} \|\Phi\|_{L^\infty(D)}\big).$$

In particular, if D is a ball centred in x_0, if γ, $\widetilde{\gamma}$, μ and $\widetilde{\mu}$ are scalar and Hölder continuous functions and if Φ is Hölder continuous, then for some $\beta > 0$

$$\frac{1}{|D|} \int_D (\widetilde{\gamma} - \gamma) E_D \cdot \Phi \, dx = \frac{3\gamma(x_0)(\widetilde{\gamma}(x_0) - \gamma(x_0))}{2\gamma(x_0) + \widetilde{\gamma}(x_0)} E(x_0) \cdot \Phi(x_0) + O(|D|^\beta),$$

$$\frac{1}{|D|} \int_D (\widetilde{\mu} - \mu) H_D \cdot \Phi \, dx = \frac{3\mu(x_0)(\widetilde{\mu}(x_0) - \mu(x_0))}{2\mu(x_0) + \widetilde{\mu}(x_0)} H(x_0) \cdot \Phi(x_0) + O(|D|^\beta).$$

Proof. — By Theorem 4.4, write

$$\int_D (\widetilde{\gamma} - \gamma) E_D \cdot \Phi \, dx = \int_D (\widetilde{\gamma} - \gamma)(I_3 + P_D^\gamma) E \cdot \Phi + (\widetilde{\gamma} - \gamma) \widetilde{R}_D^E \cdot \Phi \, dx.$$

Applying the Cauchy–Schwarz inequality, we find

$$\left| \int_D (\widetilde{\gamma} - \gamma) \widetilde{R}_D^E \cdot \Phi \, dx \right| \leq \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}} \cdot \left\| \widetilde{R}_D^E \right\|_{L^2(\Omega)} \cdot \|\Phi\|_{L^\infty(D)}$$

$$= O\big(|D|^{1+\delta} \|\Phi\|_{L^\infty(D)}\big).$$

The second identity is similar. $\qquad\square$

We conclude this section by observing that by considering the particular case $\omega = 0$, we obtain the expansion related to the conductivity equation. This case has been widely studied, see *e.g.* [**30**], [**79**], [**73**], [**171**].

Theorem 4.7. — *Let Ω, Ω_0 and D be as above. Let $\sigma \in W^{1,p}(\overline{\Omega}; \mathbb{R}^{3\times3})$ and $\widetilde{\sigma} \in L^\infty(\Omega; \mathbb{R}^{3\times3})$ satisfy (4.4) for some $p > 3$, and set*

$$\sigma_D = \sigma + (\widetilde{\sigma} - \sigma) \mathbb{1}_D.$$

For $v \in H^1(\Omega; \mathbb{R})$ and $f \in L^p(\Omega; \mathbb{R})$, let $u, u_D \in H^1(\Omega; \mathbb{R})$ be the solutions to

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = f & in \ \Omega, \\ u = v & on \ \partial\Omega, \end{cases} \qquad \begin{cases} -\operatorname{div}(\sigma_D \nabla u_D) = f & in \ \Omega, \\ u_D = v & on \ \partial\Omega. \end{cases}$$

Then we have the expansion

$$\nabla u_D = (I_3 + P_D^\sigma) \nabla u + r_D \quad in \ \Omega,$$

where the remainder term r_D is bounded by

$$\|r_D\|_{L^2(\Omega_0)} \leq C |D|^{\frac{1}{2}+\delta} \|\widetilde{\sigma} - \sigma\|_{L^\infty(D)} \big(\|v\|_{H^1(\Omega)} + \|f\|_{L^p(\Omega)} \big)$$

for some $\delta > 0$ depending only on Ω, Λ and p and some $C > 0$ depending only on Ω, Ω_0, Λ and $\|\sigma\|_{W^{1,p}(\Omega)}$. In particular, for every $\Phi \in L^\infty(\Omega_0; \mathbb{C}^3)$ we have

$$\int_D (\widetilde{\sigma} - \sigma) \nabla u_D \cdot \Phi \, dx = \int_D (\widetilde{\sigma} - \sigma)(I_3 + P_D^\sigma) \nabla u \cdot \Phi \, dx + O\big(|D|^{1+\delta} \|\Phi\|_{L^\infty(D)}\big),$$

and, if D is a ball centred in x_0, if σ and $\widetilde{\sigma}$ are scalar and Hölder continuous functions and if Φ is Hölder continuous, then for some $\beta > 0$

$$\frac{1}{|D|} \int_D (\widetilde{\sigma} - \sigma) \nabla u_D \cdot \Phi \, dx = \frac{3\sigma(x_0)(\widetilde{\sigma}(x_0) - \sigma(x_0))}{2\sigma(x_0) + \widetilde{\sigma}(x_0)} \nabla u(x_0) \cdot \Phi(x_0) + O\big(|D|^\beta\big).$$

Proof. — In view of Proposition 3.19 and Remark 3.20, this result is a consequence of Theorem 4.4 and Corollary 4.6, with the identifications $E = \nabla u$ and $E_D = \nabla u_D$. $\qquad\square$

4.4. Proofs of Theorems 4.2 and 4.4

In order to prove these results, we consider the difference between the perturbed and the unperturbed (or background) problems, namely

$$(W_D, Q_D) = (E_D - E, H_D - H) \in H(\mathrm{curl}, \Omega)^2.$$

The pair (W_D, Q_D) satisfies

$$(4.19) \quad \begin{cases} \mathrm{curl}\, W_D = i\omega\mu_D Q_D + i\omega \mathbb{1}_D(\mu_D - \mu)H & \text{in } \Omega, \\ \mathrm{curl}\, Q_D = -i\gamma_D W_D - i\mathbb{1}_D(\gamma_D - \gamma)E & \text{in } \Omega, \\ W_D \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Problem (4.19) has two source terms, generated by E and H. We first notice that at leading order the two fields are decoupled. Write

$$(W_D, Q_D) = (W_D^E, Q_D^E) + (W_D^H, Q_D^H)$$

where

$$(4.20) \quad \begin{cases} \mathrm{curl}\, W_D^E = i\omega\mu_D Q_D^E & \text{in } \Omega, \\ \mathrm{curl}\, Q_D^E = -i\gamma_D W_D^E - i\mathbb{1}_D(\gamma_D - \gamma)E & \text{in } \Omega, \\ W_D^E \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(4.21) \quad \begin{cases} \operatorname{curl} W_D^H = i\omega\mu_D Q_D^H + i\omega \, \mathbb{1}_D (\mu_D - \mu) H & \text{in } \Omega, \\ \operatorname{curl} Q_D^H = -i\gamma_D W_D^H & \text{in } \Omega, \\ W_D^H \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The following result shows that W_D can be identified with W_D^E, whereas Q_D can be identified with Q_D^H as a first approximation in $|D|$.

Lemma 4.8. — *Under the assumptions of Theorem 4.2, there exists a constant $C > 0$ depending only on Ω, Ω_0, Λ, C_0, $\|(\mu,\gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$ and $|\omega|$, and a constant $\delta > 0$ depending only on Ω and Λ such that*

$$\|(W_D, Q_D)\|_{H(\operatorname{curl},\Omega)^2} \leq C|D|^{\frac{1}{2}} \cdot \big\| \max \big(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma| \big) \big\|_{L^\infty(D)} \cdot M_\alpha,$$

and

$$(4.22) \quad \begin{cases} \|W_D - W_D^E\|_{L^2(\Omega)^2} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\widetilde{\mu} - \mu\|_{L^\infty(D)} \cdot M_\alpha, \\ \|Q_D - Q_D^H\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot M_\alpha. \end{cases}$$

Proof. — With an abuse of notation, several positive constants depending only on Ω, Ω_0, Λ, C_0, $\|(\mu,\gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$ and $|\omega|$ will be denoted by C.

By Lemma 4.1 applied to (4.19), we have

$$\|(W_D, Q_D)\|_{H(\operatorname{curl},\Omega)} \leq C \big(\|\mathbb{1}_D (\widetilde{\mu} - \mu) H\|_{L^2(\Omega)} + \|\mathbb{1}_D (\widetilde{\gamma} - \gamma) E\|_{L^2(\Omega)} \big).$$

In view of (4.6) and (4.15), Theorem 3.10 yields $(E, H) \in C^{0,\alpha}(\overline{\Omega_0}; \mathbb{C}^3)^2$ and

$$(4.23) \qquad \|(E, H)\|_{C^{0,\alpha}(\overline{\Omega_0})^2} \leq C \big(\|\varphi\|_{H(\operatorname{curl},\Omega)} + \|(J, K)\|_{C^{0,\alpha}(\overline{\Omega})^2} \big) = C M_\alpha.$$

Using Hölder's inequality, these two estimates imply

$$\|(W_D, Q_D)\|_{H(\operatorname{curl},\Omega)^2} \leq C|D|^{\frac{1}{2}} \cdot \big\| \max \big(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma| \big) \big\|_{L^\infty(D)} M_\alpha,$$

as announced.

We prove (4.22) using a variant of the so-called Aubin–Nitsche (or Céa) duality argument in numerical analysis. Consider the adjoint problem associated

to (4.7), with respect to $L^2(\Omega; \mathbb{C}^3)^2$, namely

$$(4.24) \quad \begin{cases} \operatorname{curl} X_D = \mathrm{i}\bar{\omega}\bar{\mu}_D T_D + B & \text{in } \Omega, \\ \operatorname{curl} T_D = -\mathrm{i}\bar{\gamma}_D X_D + A & \text{in } \Omega, \\ X_D \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

with $A, B \in L^2(\Omega; \mathbb{C}^3)$. Note that problem (4.24) is well posed whenever problem (4.7) is well posed (Remark 3.16), and so by Lemma 4.1 we have

$$(4.25) \qquad \|(X_D, T_D)\|_{H(\operatorname{curl},\Omega)^2} \le C \|(A, B)\|_{L^2(\Omega)^2}.$$

By an integration by parts, we have the duality identity

$$(4.26) \quad \int_\Omega E_D \cdot \bar{A}\,\mathrm{d}x - \int_\Omega H_D \cdot \bar{B}\,\mathrm{d}x$$
$$= \int_\Omega K \cdot \overline{T}_D\,\mathrm{d}x - \int_\Omega J \cdot \overline{X}_D\,\mathrm{d}x - \int_{\partial\Omega} (\varphi \times \nu) \cdot \overline{T}_D\,\mathrm{d}\sigma,$$

for any solution $(E_D, H_D) \in H(\operatorname{curl}; \Omega)^2$ of (4.7) and any solution $(T_D, X_D) \in H(\operatorname{curl}; \Omega)^2$ of (4.24). If we apply this identity to (4.20), that is, $K = 0$ and $J = \mathrm{i}\mathbb{1}_D(\gamma_D - \gamma)E$, and (4.24) with $A = 0$ and $B = Q_D^E$, we obtain

$$(4.27) \qquad \|Q_D^E\|_{L^2(\Omega)}^2 = -\int_\Omega \mathrm{i}\mathbb{1}_D(\gamma_D - \gamma)E \cdot \overline{X}_D\,\mathrm{d}x.$$

In addition, (4.25) and Theorem 3.14 applied to (4.24) yield

$$\|X_D\|_{L^{2+\delta}(\Omega_0)} \le C\|Q_D^E\|_{L^2(\Omega)}$$

for some $\delta > 0$ depending only Λ and Ω as introduced before. Combining this estimate with (4.27), and using Hölder's inequality, we obtain

$$\|Q_D^E\|_{L^2(\Omega)}^2 \le C\|\gamma_D - \gamma\|_{L^\infty(D)} \cdot \|E\|_{L^2(D)} \cdot \|Q_D^E\|_{L^2(\Omega)} \cdot |D|^{\frac{\delta}{4+2\delta}},$$

which, in view of (4.23), provides

$$\|Q_D^E\|_{L^2(\Omega)} \le C|D|^{\frac{1}{2}+\frac{\delta}{4+2\delta}} \cdot \|\gamma_D - \gamma\|_{L^\infty(D)} \cdot M_\alpha,$$

as desired. Using the same method, we apply the identity (4.26) to (4.21) and (4.24) with $A = W_D^H$ and $B = 0$ and we obtain

$$\|W_D^H\|_{L^2(\Omega)} \le C|D|^{\frac{1}{2}+\frac{\delta}{4+2\delta}} \cdot \|\mu_D - \mu\|_{L^\infty(D)} \cdot M_\alpha. \qquad \square$$

It turns out that the dominant parts of W_D^E and Q_D^H are their non rotational components, as already anticipated in Remark 3.15.

Lemma 4.9. — *Under the assumptions of Lemma 4.8, and with the same notations, there exists $C > 0$ depending only on Ω, Ω_0, Λ, C_0, $\|(\mu, \gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$ and $|\omega|$ such that*

$$\|E_D - E - \nabla p_D^E\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\max(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma|)\|_{L^\infty(D)} \cdot M_\alpha,$$

$$\|H_D - H - \nabla p_D^H\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\max(|\widetilde{\mu} - \mu|, |\widetilde{\gamma} - \gamma|)\|_{L^\infty(D)} \cdot M_\alpha,$$

where p_D^E and p_D^H satisfy (4.13) and (4.14).

Proof. — With an abuse of notation, several positive constants depending only on Ω, Ω_0, Λ, C_0, $\|(\mu, \gamma)\|_{C^{0,\alpha}(\overline{\Omega})^2}$ and $|\omega|$ will be denoted by C.

Decompose W_D^E into its gradient and rotational parts as described in Lemma 3.7 (Helmholtz decomposition), namely

$$W_D^E = \nabla q_D^E + \mathrm{curl}\, \Phi_D^E,$$

where $q_D^E \in H_0^1(\Omega; \mathbb{C})$ and $\Phi_D^E \in H^1(\Omega; \mathbb{C}^3)$. Setting $\Psi_D^E = \mathrm{curl}\, \Phi_D^E$, from the definition of W_D^E (4.20) and the fact that q_D^E is constant on $\partial\Omega$ we find

$$\begin{cases} \mathrm{curl}\, \Psi_D^E = i\omega\mu_D Q_D^E & \text{in } \Omega, \\ \mathrm{div}\, \Psi_D^E = 0 & \text{in } \Omega, \\ \Psi_D^E \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

which by Lemmata 3.6 and 4.8 implies

$$(4.28) \qquad \|\mathrm{curl}\, \Phi_D^E\|_{H^1(\Omega)} \leq C\|Q_D^E\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\gamma_D - \gamma\|_{L^\infty(D)} \cdot M_\alpha.$$

Taking the divergence of the second identity in (4.20) and recalling the definition of p_D^E we find

$$-\mathrm{div}(\gamma_D \nabla q_D^E) = \mathrm{div}\left(\mathbb{1}_D(\gamma_D - \gamma)E + \gamma_D \,\mathrm{curl}\, \Phi_D^E\right) \quad \text{in } \Omega,$$

$$-\mathrm{div}\left(\gamma_D \nabla p_D^E\right) = \mathrm{div}(\mathbb{1}_D(\gamma_D - \gamma)E) \quad \text{in } \Omega,$$

therefore

$$\begin{cases} -\mathrm{div}\left(\gamma_D \nabla(q_D^E - p_D^E)\right) = \mathrm{div}(\gamma_D \,\mathrm{curl}\, \Phi_D^E) & \text{in } \Omega, \\ q_D^E - p_D^E = 0 & \text{on } \partial\Omega. \end{cases}$$

As a consequence, by (4.28) we obtain

$$\|\nabla q_D^E - \nabla p_D^E\|_{L^2(\Omega)} \leq C\|\operatorname{curl}\Phi_D^E\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\gamma_D - \gamma\|_{L^\infty(D)} \cdot M_\alpha.$$

Thus, the conclusion follows by (4.22) and (4.28).

The justification of the expansion of H is entirely similar, and only the study of the boundary conditions is different. More precisely, write

$$Q_D^H = \nabla q_D^H + \operatorname{curl}\Phi_D^H,$$

with $q_D^H \in H^1(\Omega;\mathbb{C})$ and $\Phi_D^H \in H^1(\Omega;\mathbb{C}^3)$ such that $\Phi_D^H \times \nu = 0$ on $\partial\Omega$. Setting $\Psi_D^H = \operatorname{curl}\Phi_D^H$, from the definition of Q_D^H in (4.21) we find

$$\begin{cases} \operatorname{curl}\Psi_D^H = -\mathrm{i}\gamma_D W_D^H & \text{in } \Omega, \\ \operatorname{div}\Psi_D^H = 0 & \text{in } \Omega, \\ \Psi_D^H \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where the boundary condition follows from [**163**, equation (3.52)] and the fact that $\Phi_D^H \times \nu = 0$ on $\partial\Omega$, since $\Psi_D^H \cdot \nu = \operatorname{div}_{\partial\Omega}(\Phi_D^H \times \nu) = 0$ on $\partial\Omega$. Lemmata 3.6 and 4.8 yield

$$(4.29) \qquad \|\operatorname{curl}\Phi_D^H\|_{H^1(\Omega)} \leq C\|W_D^H\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\mu_D - \mu\|_{L^\infty(D)} \cdot M_\alpha.$$

Using the first identity in (4.21) and the fact that $\mu_D Q_D^H \cdot \nu = 0$ on $\partial\Omega$ by [**163**, equation (3.52)], we find

$$\begin{cases} -\operatorname{div}\left(\mu_D \nabla(q_D^H - p_D^H)\right) = \operatorname{div}(\mu_D \operatorname{curl}\Phi_D^H) & \text{in } \Omega, \\ -\mu_D \nabla(q_D^H - p_D^H) \cdot \nu = \mu_D \operatorname{curl}\Phi_D^H \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

As a consequence, by (4.29) we obtain

$$\|\nabla q_D^H - \nabla p_D^H\|_{L^2(\Omega)} \leq C\|\operatorname{curl}\Phi_D^H\|_{L^2(\Omega)} \leq C|D|^{\frac{1}{2}+\delta} \cdot \|\mu_D - \mu\|_{L^\infty(D)} \cdot M_\alpha.$$

Thus, the conclusion follows by (4.22) and (4.29). $\qquad\qquad\square$

Lemma 4.9 was the missing piece in the the proof of Theorem 4.2. To conclude the proof of Theorem 4.4, we need to show how regularity allows for a separation of scales. This is the purpose of this final lemma.

Lemma 4.10. — *Under the assumptions of Theorem 4.4, there exists $\delta' > 0$ depending only on δ and p such that*

$$\|\nabla p_D^E - P_D^\gamma E\| \leq C \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\delta'} \cdot M_p,$$

$$\|\nabla p_D^H - P_D^\mu H\| \leq C \|\widetilde{\mu} - \mu\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\delta'} \cdot M_p,$$

for some $C > 0$ depending only on Ω, Ω_0, Λ, C_0, $\|(\mu,\gamma)\|_{W^{1,p}(\Omega)^2}$ and $|\omega|$.

Proof. — Several positive constants depending only on Ω, Ω_0, Λ, C_0, $|\omega|$ and $\|(\mu,\gamma)\|_{W^{1,p}(\Omega)^2}$ will be denoted by C. We use Einstein summation convention (repeated indices are implicitly summed over).

Let Ω_1 be a smooth subdomain such that $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Theorem 3.12 yields $(E, H) \in W^{1,p}(\Omega_1; \mathbb{C}^3)^2$ and

$$(4.30) \qquad \|(E, H)\|_{W^{1,p}(\Omega_1)^2} \leq C \big(\|\varphi\|_{H(\mathrm{curl},\Omega)} + \|(J, K)\|_{W^{1,p}(\mathrm{div},\Omega)^2} \big) = C M_p.$$

Testing (4.16) against $\overline{p_\gamma^k}$, integrating by parts and using ellipticity we obtain

$$(4.31) \qquad \|\nabla p_\gamma^k\|_{L^2(\Omega)} \leq C \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}}.$$

While this estimate is optimal for the gradient, it can be improved for the potential itself, using the Aubin–Nitsche duality argument. In detail, let z in $H_0^1(\Omega; \mathbb{C})$ be the solution of

$$(4.32) \qquad -\operatorname{div}(\overline{\gamma}_D \nabla z) = p_\gamma^k \quad \text{in } \Omega.$$

Thanks to Theorem 3.5 (with $q = 6$ and $p = \frac{3q}{q+3} = 2$), we have that ∇z enjoys higher integrability in Ω_0, namely

$$(4.33) \qquad \|\nabla z\|_{L^{2+\delta}(\Omega_0)} \leq C \|p_\gamma^k\|_{L^2(\Omega)}.$$

Moreover, testing (4.32) against $\overline{p_\gamma^k}$ and comparing it with (4.16) tested against $\bar{z}$, we obtain

$$\|p_\gamma^k\|_{L^2(\Omega)}^2 = \int_\Omega \nabla z \cdot \overline{\gamma}_D \nabla \overline{p_\gamma^k} \, dx = \int_{\Omega_0} \nabla z \cdot \mathbb{1}_D (\overline{\gamma} - \overline{\gamma}_D) \mathbf{e}_k \, dx.$$

Thanks to Hölder's inequality, this yields

$$\|p_\gamma^k\|_{L^2(\Omega)}^2 \leq \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2} + \frac{\delta}{4+2\delta}} \cdot \|\nabla z\|_{L^{2+\delta}(\Omega_0)}$$
$$\leq C\|p_\gamma^k\|_{L^2(\Omega)} \cdot \|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2} + \frac{\delta}{4+2\delta}},$$

using (4.33) in the second step. We have obtained that

$$(4.34) \qquad \|p_\gamma^k\|_{L^2(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2} + \frac{\delta}{2\delta+4}}.$$

The Gagliardo–Nirenberg–Sobolev inequality (see *e.g.* [**95**, Chapter 5.6]) then shows that (4.31) and (4.34) imply in particular that

$$(4.35) \qquad \|p_\gamma^k\|_{L^{\frac{2p}{p-2}}(\Omega)} \leq C\|\nabla p_\gamma^k\|_{L^2(\Omega)}^{\frac{3}{p}} \cdot \|p_\gamma^k\|_{L^2(\Omega)}^{1-\frac{3}{p}} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\delta'},$$

where $\delta' = \frac{\delta}{4+2\delta}(1 - \frac{3}{p})$. The same arguments when applied to p_D^E show that

$$(4.36) \qquad \|\nabla p_D^E\|_{L^2(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}} \cdot \|E\|_{L^\infty(\Omega_1)}$$
$$\leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}} \cdot M_p,$$

and, in turn,

$$(4.37) \qquad \|p_D^E\|_{L^{\frac{2p}{p-2}}(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\delta'} \cdot M_p.$$

We shall now make use of (4.35) and (4.37) to conclude.

Let $\chi \in C^\infty(\Omega)$ be a cut-off function such that $\chi = 1$ in Ω_0 and $\operatorname{supp}\chi \subseteq \Omega_1$. We wish to show that

$$(4.38) \qquad \|\chi(\nabla p_D^E - P_D^\gamma E)\|_{L^2(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\frac{\delta'}{2}} \cdot M_p,$$

as it establishes our claim with respect to the electric field. The estimate on the magnetic field may be derived in a similar way, the only relevant difference lies in the Neumann boundary conditions in place of the Dirichlet boundary conditions, but it does not cause additional difficulties. We will focus therefore on (4.38), and more precisely we shall show that

$$(4.39) \qquad w = p_D^E - p_\gamma^k E_k \in H_0^1(\Omega),$$

satisfies

$$(4.40) \qquad \|\chi \nabla w\|_{L^2(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\frac{\delta'}{2}} \cdot M_p.$$

Indeed (4.40) implies (4.38), as thanks to (4.30) and (4.35) we have

$$\|\chi(\nabla w - (\nabla p_D^E - P_D^\gamma E))\|_{L^2(\Omega)} = \|\chi p_\gamma^k \nabla E_k\|_{L^2(\Omega)}$$

$$\leq C\|\nabla E_k\|_{L^p(\Omega_1)} \cdot \|p_\gamma^k\|_{L^{2p/(p-2)}(\Omega)} \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot |D|^{\frac{1}{2}+\delta'} \cdot M_p.$$

Thanks to Hölder's inequality, and estimates (4.35) and (4.37) we have

$$(4.41) \qquad \|w\|_{L^{2p/(p-2)}(\Omega_1)} \leq \|p_D^E\|_{L^{2p/(p-2)}(\Omega)} + \|p_\gamma^k\|_{L^{2p/(p-2)}(\Omega)}\|E_k\|_{L^\infty(\Omega_1)}$$

$$\leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot M_p \cdot |D|^{\frac{1}{2}+\delta'},$$

whereas from (4.31) and (4.36) we obtain

$$(4.42) \quad \|\nabla w\|_{L^2(\Omega_1)} \leq \|\nabla p_D^E\|_{L^2(\Omega)} + \|\nabla p_\gamma^k\|_{L^2(\Omega)} \cdot \|E_k\|_{L^\infty(\Omega_1)} + \|p_\gamma^k \nabla E_k\|_{L^2(\Omega_1)}$$

$$\leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)} \cdot M_p \cdot |D|^{\frac{1}{2}}.$$

Using the ellipticity of γ_D, we find

$$\|\chi \nabla w\|_{L^2(\Omega)}^2 \leq 2\|\nabla(\chi w)\|_{L^2(\Omega)}^2 + 2\|w\nabla\chi\|_{L^2(\Omega)}^2$$

$$\leq C\left(\left|\int_\Omega \gamma_D \nabla\overline{\chi w} \cdot \nabla(\chi w)\,\mathrm{d}x\right| + \|w\nabla\chi\|_{L^2(\Omega)}^2\right)$$

$$\leq C\left(\left|\int_\Omega \gamma_D \nabla\overline{w} \cdot \nabla(\chi^2 w)\,\mathrm{d}x\right| + \|w\nabla\chi\|_{L^2(\Omega)}^2 + \|\chi\nabla w\|_{L^2(\Omega)}\|w\nabla\chi\|_{L^2(\Omega)}\right),$$

In other words, we have

$$\|\chi\nabla w\|_{L^2(\Omega)} \leq C\left(\left|\int_\Omega \gamma_D \nabla\overline{w} \cdot \nabla(\chi^2 w)\,\mathrm{d}x\right|^{\frac{1}{2}} + \|w\nabla\chi\|_{L^2(\Omega)}\right).$$

Thus, thanks to (4.41), we find that proving (4.40) reduces to showing that

$$(4.43) \qquad \left|\int_\Omega \gamma_D \nabla\overline{w} \cdot \nabla(\chi^2 w)\,\mathrm{d}x\right| \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)}^2 \cdot M_p^2 \cdot |D|^{1+\delta'}.$$

Using (4.13) and (4.16) we find

$$
-\int_\Omega \gamma_D \nabla w \cdot \nabla(\chi^2 \overline{w})\,\mathrm{d}x
$$

$$
= \int_\Omega \operatorname{div}(\gamma_D \nabla p_D^E)\chi^2 \overline{w} - \operatorname{div}(\gamma_D \nabla p_\gamma^k)E_k \chi^2 \overline{w}\,\mathrm{d}x
$$

$$
+ \int_\Omega \gamma_D p_\gamma^k \nabla E_k \cdot \nabla(\chi^2 \overline{w})\,\mathrm{d}x - \int_\Omega \gamma_D \nabla E_k \cdot \nabla p_\gamma^k(\chi^2 \overline{w})\,\mathrm{d}x
$$

$$
= \int_\Omega -\chi^2 \overline{w}\,\operatorname{div}(\mathbb{1}_D(\gamma_D - \gamma)E_k \mathbf{e}_k)\,\mathrm{d}x
$$

$$
+ \int_\Omega \chi^2 \overline{w}E_k \operatorname{div}(\mathbb{1}_D(\gamma_D - \gamma)\mathbf{e}_k)\,\mathrm{d}x
$$

$$
+ \int_\Omega \gamma_D p_\gamma^k \nabla E_k \cdot \nabla(\chi^2 \overline{w})\,\mathrm{d}x - \int_\Omega \gamma_D \nabla E_k \cdot \nabla p_\gamma^k(\chi^2 \overline{w})\,\mathrm{d}x
$$

$$
= \int_\Omega -\chi^2 \overline{w}\,\mathbb{1}_D(\gamma_D - \gamma)\mathbf{e}_k \nabla E_k\,\mathrm{d}x
$$

$$
+ \int_\Omega \gamma_D p_\gamma^k \nabla E_k \cdot \nabla(\chi^2 \overline{w})\,\mathrm{d}x - \int_\Omega \gamma_D \nabla E_k \cdot \nabla p_\gamma^k(\chi^2 \overline{w})\,\mathrm{d}x.
$$

To conclude we bound each of the right-hand side terms. Using Hölder's inequality, (4.30) and (4.41), we find

$$
\left| \int_\Omega \chi^2 \overline{w}\,\mathbb{1}_D(\gamma_D - \gamma)\mathbf{e}_k \nabla E_k\,\mathrm{d}x \right| \leq C\|\widetilde{\gamma} - \gamma\|_{L^2(D)} \cdot \|w\|_{L^{2p/(p-2)}(\Omega_1)} \|\nabla E_k\|_{L^p(\Omega_1)}
$$

$$
\leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)}^2 \cdot M_p^2 \cdot |D|^{1+\delta'},
$$

and this bound agrees with (4.43). Using Hölder's inequality, (4.30), (4.35), (4.41) and (4.42), we obtain

$$
\left| \int_\Omega \gamma_D p_\gamma^k \nabla E_k \cdot \nabla(\chi^2 \overline{w})\,\mathrm{d}x \right| \leq C\|p_\gamma^k\|_{L^{2p/(p-2)}(\Omega)} \cdot \|\nabla E_k\|_{L^p(\Omega_1)} \cdot \|w\|_{H^1(\Omega_1)}
$$

$$
\leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)}^2 \cdot M_p^2 \cdot |D|^{1+\delta'},
$$

which is also the expected bound. Finally, using Hölder's inequality, (4.30), (4.31) and (4.41) we find

$$
\left| \int_\Omega \gamma_D \nabla E_k \cdot \nabla p_\gamma^k(\chi^2 \overline{w})\,\mathrm{d}x \right| \leq C\|\widetilde{\gamma} - \gamma\|_{L^\infty(D)}^2 \cdot M_p^2 \cdot |D|^{1+\delta'}
$$

which concludes the proof of (4.43). □

CHAPTER 5

A CASE STUDY OF SCATTERING ESTIMATES

5.1. Introduction

In Chapter 2, we mentioned that observability properties for the wave equation in inhomogeneous media were connected to the regularity properties of the parameters. In Chapter 3, we showed that some regularity of the coefficients were required to derive $W^{m,p}$ estimates for the electromagnetic fields in general. In practical applications, the observed medium is not perfect: small defects may occur. In Chapter 4, we detailed how the first order terms of the expansion in terms of the volume of such defects are obtained, for a given frequency (or more generally for frequencies within a give range, away from the eigenvalues of the boundary value problem). The first order expansion obtained for the perturbation of the electromagnetic energy was a product of the difference in the coefficients, that is, $\widetilde{\varepsilon} - \varepsilon$ for the electric field and $\widetilde{\mu} - \mu$ for the magnetic field, the frequency ω, and the volume of the inclusion $|D|$. It is natural to wonder what the limits of such asymptotic regimes are, namely when ω or $\widetilde{\varepsilon} - \varepsilon$ are very large compared to the volume. In the case of a diametrically small inclusion and when $\omega = 0$, it is known that the corresponding expansions are in fact uniform with respect to the contrast [171].

An educated guess, extrapolating from the results of the previous chapter, is that if the contrast between the defects and the background medium, the size of the inclusion, and the time-dependent oscillation of the fields are sufficiently

small, the effect of the defects is negligible. The purpose of this chapter is to quantify precisely this statement on a simple model, namely the following time-harmonic Helmholtz equation

$$\Delta u + \omega^2 q^2 u = 0 \quad \text{in } \mathbb{R}^d,$$

with $d = 2$ or 3, where q takes two values,

$$q(x) = \begin{cases} a & \text{when } |x| < \varepsilon, \\ 1 & \text{otherwise}, \end{cases}$$

with $a, \varepsilon > 0$. This model can be derived from the acoustic wave equation: writing $c = q^{-1}$ and $U(x,t) = u(x)\,e^{i\omega t}$, U is a solution of

$$\partial_{tt} U - c^2 \Delta U = 0 \quad \text{in } \mathbb{R}^d.$$

Alternatively, in dimension 2, it also corresponds to the transverse electric mode of Maxwell's system for an isotropic dielectric parameter $\varepsilon = q^2$ in a magnetically homogeneous medium.

We are interested in scattering estimates, that is, in evaluating how much u departs from the solution of the homogeneous problem u^i for $q \equiv 1$ everywhere (no defects), when u and u^i are close to each other at infinity. The scattering wave equation writes

(5.1) $$\begin{cases} \Delta u + \omega^2 q^2 u = 0 & \text{in } \mathbb{R}^d, \\ u - u^i & \text{satisfies the radiation condition.} \end{cases}$$

The last statement of (5.1) precisely means that, writing $u = u^i + u^s$ for $|x| > \varepsilon$, u^s satisfies

(5.2) $$\lim_{r \to \infty} r^{\frac{1}{2}(d-1)} \left(\frac{\partial u^s}{\partial r}(x) - i\omega u^s(x) \right) = 0 \quad \text{with } r = |x|,$$

and this condition holds uniformly for every direction $x/|x| \in \mathbb{S}^{d-1}$. This condition is known as the Sommerfeld radiation condition, and it characterises uniquely the radiative nature of the scattered field u^s, see [82], [170], [122] for a proof of the well-posedness of (5.1)–(5.2). The solution u is the unique weak solution in $H^1_{\text{loc}}(\mathbb{R}^d)$.

Scattering estimates correspond to free space problems, when boundary conditions are not present, or are deemed far enough away so that they do

not affect the problem at hand. In the present case, the Helmholtz equation presents the added advantage of having no eigenmodes (or resonances), thus any $\omega > 0$ may be considered.

It is very well known that, for fixed ω and a, the scattered field decays as $|x| \to \infty$; for example for an incident field $u^i = \exp(i\omega\xi \cdot x)$, with $\xi \in \mathbb{S}^{d-1}$, there holds [82], [122]

$$u = u^i + \frac{\exp(i\omega|x|)}{|x|^{\frac{1}{2}(d-1)}} u_\infty \left(\frac{x}{|x|}, \xi, \omega \right) + O(|x|^{-\frac{1}{2}(d+1)}),$$

where u_∞, the legacy of the inclusion at infinity, is called the far field pattern, or scattering amplitude. The dependence of this asymptotic on the contrast a and on the frequency of the incident field is not straightforward. It is also clear that such an expansion is of little use in the near field, that is, close to the inclusion. The purpose of this chapter is to provide quantitative estimates for any contrast, any frequency and at arbitrary distances for $u - u^i$. In presence of a lossy layer (when q has an imaginary part), estimates for all frequencies have been obtained in [172].

This is very difficult in general, but the particular case we are considering has been studied for well over a century: the Bessel functions are solutions of this problem obtained by separation of variables in polar (or spherical) coordinates. These functions have been studied extensively, and their asymptotic properties are well known [176, Chapter 10]. Much less is known regarding uniform estimates of these functions (historically, the emphasis was centred around their numerical approximation, which has become less crucial in modern times): this is still the topic of on-going investigations. While our analysis uses these functions in an essential manner, and most of this chapter is centred around properties of Bessel functions, they do not appear in our final results, which do not require any knowledge of the shibboleth of Bessel functions to be stated.

The results presented in this chapter mostly stem from [75], [76], [72] and [122], to which we refer the readers for additional and more complete results.

5.2. Main results

In what follows, $u^i \in H^1_{loc}(\mathbb{R}^d; \mathbb{C})$ will refer to a solution of

$$\Delta u^i + \omega^2 u^i = 0 \quad \text{in } \mathbb{R}^d,$$

whereas $u^s \in H^1_{loc}(\mathbb{R}^d; \mathbb{C})$ is defined as the unique solution of

$$(5.3) \quad \begin{cases} \Delta(u^i + u^s) + \omega^2 q^2 (u^i + u^s) = 0 & \text{in } \mathbb{R}^d, \\ \lim_{r \to \infty} r^{\frac{1}{2}(d-1)} \left(\dfrac{\partial u^s}{\partial r}(x) - i\omega u^s(x) \right) = 0 \\ \qquad\qquad\qquad\qquad\qquad \text{uniformly for all } x/|x| \in \mathbb{S}^{d-1}. \end{cases}$$

The definition of the norms used is recalled in Section 5.4. Given a Banach space H of functions defined on $\mathbb{S}^{d-1}$ and a function f defined on $\mathbb{R}^d$, with an abuse of notation we shall denote

$$\left\| f(|x| = R) \right\|_H := \| f_R \|_H, \quad R > 0,$$

where f_R is the function defined by $f_R(x) = f(Rx)$ for $x \in \mathbb{S}^{d-1}$.

Our first result quantifies the regime for which local estimates hold.

Theorem 5.1. — *For all* $s \geq 0$, $a > 0$, $R \geq \varepsilon$ *and* $\omega > 0$ *such that* $\max(a, 1)\omega\varepsilon \leq 1$, *there holds*

$$\left\| u^s(|x| = R) \right\|_{H^{s+1/3}_*(\mathbb{S}^{d-1})} \leq 3|a - 1|\omega\varepsilon \left(\frac{\varepsilon}{R} \right)^{\frac{1}{2}(d-1)} \cdot \left\| u^i(|x| = \varepsilon) \right\|_{H^s_*(\mathbb{S}^{d-1})}.$$

This result shows the local character of the perturbation for moderate frequencies, that is $\max(a, 1)\omega \leq \varepsilon^{-1}$: the perturbation is controlled by the norm of the incident field on the obstacle. Note that the contrasting extremes, $a \to 0$ or a large, lead to very different estimates: the norm of the scattered field is controlled uniformly for $0 < a < 1$, and for all frequencies lower that ε^{-1}. This is consistent with general Morrey–Campanato estimates established for Helmholtz equation in [**180**] with a variable index $q(x)$. Such estimates require a growth condition on q to hold: in particular, they hold when $\nabla q \cdot x$ is non-negative. When $a < 1$, the discontinuous index $q(r)$ which jumps from a to 1 at $r = \varepsilon$ is the limit of a sequence of smooth monotonously radially increasing indexes $q_n(r)$ equal to a until $\varepsilon - \frac{1}{n}$ and 1 when $r \geq \varepsilon$. On the other hand,

when $a > 1$, [**180**] gives no insight on what the estimate should be, while Theorem 5.1 introduces a restriction on the frequency depending on the contrast factor a.

While this book was being reviewed, we were made aware of [**159**], where the case when $a \leq 1$ is addressed, for star shaped domains, using a method based on Morawetz multipliers.

Our second result reads as follows.

Theorem 5.2. — *For any $s \geq 0$, $a > 0$, $R \geq \max(1, a)\varepsilon$ and $\omega > 0$ there holds*

$$\left\|u^s(|x| = R)\right\|_{H^s_*(\mathbb{S}^{d-1})}$$
$$\leq 3\left(\frac{\max(a, 1)\varepsilon}{R}\right)^{\frac{1}{2}(d-1)} \left(1 + \omega\varepsilon\max(a, 1)\right) \sup_{0 \leq \tau \leq \max(1,a)\varepsilon} \left\|u^i(|x| = \tau)\right\|_{H^s_*(\mathbb{S}^{d-1})}.$$

Remark. — For a slightly sharper estimate, see Lemma 5.12.

Example 5.3. — When $s = 0$, $d = 2$ and $u^i = \exp(i\omega x \cdot \xi)$, with $|\xi| = 1$, the estimate given by Theorem 5.1 implies that for all $R \geq \varepsilon$ and $\omega\varepsilon\max(a, 1) \leq 1$, there holds

$$\left\|u^s(|x| = R)\right\|_{H^{\frac{1}{3}}_*(\mathbb{S}^1)} \leq C|a - 1|\omega\varepsilon\sqrt{\frac{\varepsilon}{R}},$$

where C is some universal constant, whereas the estimate given by Theorem 5.2 shows that for all $R \geq \max(a, 1)\varepsilon$ and all $\omega > 0$,

$$\left\|u^s(|x| = R)\right\|_{H^0_*(\mathbb{S}^1)} \leq C\sqrt{\frac{\max(a, 1)\varepsilon}{R}}\left(1 + \omega\max(a, 1)\varepsilon\right),$$

with the same universal constant C.

Theorems 5.1 and 5.2 provide estimates for all distances $R \geq \varepsilon$ and all frequencies ω outside of the inclusion when $a \leq 1$. A near field region is not covered when $a > 1$: no estimate is provided when $\varepsilon \leq R < a\varepsilon$ and $a\omega\varepsilon > 1$. The following result illustrates why general estimates are not attainable.

Theorem 5.4. — *Let $u^i = \exp(i\omega x \cdot \xi)$, with $|\xi| = 1$. For every $a > 1$ and $\varepsilon \leq R < a\varepsilon$, there is a sequence of frequencies $(\omega_n)_{n \in \mathbb{N}}$ such that the corresponding*

scattered field u_n^s satisfy

(5.4) $\left\| u_n^s (|x| = R) \right\|_{H_*^s(\mathbb{S}^{d-1})} > C \exp \left(\tfrac{2}{5} n \left(1 - \sqrt{R/a\varepsilon} \right)^{\frac{3}{2}} \right) (1 + n^2)^{\frac{1}{2} s - C}$

for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$, where $C > 0$ is a universal constant. The sequence $(\omega_n)_{n\in\mathbb{N}}$ is not unique; there exist two intertwined sequences $(\omega_n^+)_{n\in\mathbb{N}}$ and $(\omega_n^-)_{n\in\mathbb{N}}$ satisfying

$$\cdots < \omega_n^- < \omega_n^+ < \omega_{n+1}^- < \omega_{n+1}^+ < \cdots$$

such that (5.4) holds for any $\omega_n \in (\omega_n^-, \omega_n^+)$.

This result shows that no scattering estimates such as the ones given by the previous theorems hold uniformly in ω, even for arbitrarily large negative Sobolev norms, as this only has a polynomial effect, dwarfed by the exponential dependence on the frequency. This is a quasi-resonance phenomenon, which can be shown to exist for most incident waves: the choice of an incident plane wave was made for convenience.

Remark 5.5. — This quasi-resonance phenomenon is localised around specific frequencies. If small intervals around these frequencies are removed (and such excluded frequency bands can be chosen so that their total measure tend to zero with ε) broadband estimate can be derived, see [75], [72], [76].

Further discussions of quasi-resonances, or quasi-modes, can be found in [181], [159].

For simplicity, the above estimates exclude the radial mode, which has specific features in two dimensions. A radial mode estimate can be found in [75], [72], [76], and together with Theorem 5.1 leads to the following result.

Theorem 5.6. — For any $s \in \mathbb{R}$, $R \geq \varepsilon$ and $\omega > 0$, if

$$\omega \varepsilon \max(1, a) < \min \left(\frac{1}{2}, \frac{1}{\sqrt{(3 - d) \log(\max(1, a)) + 1}} \right)$$

then

(5.5) $\left\| u^s (|x| = R) \right\|_{H^{s+1/3}(\mathbb{S}^{d-1})} \leq 3 |a - 1| \omega \varepsilon \left(\frac{\varepsilon}{R} \right)^{\frac{1}{2}(d-1)} \left\| u^i (|x| = \varepsilon) \right\|_{H^s(\mathbb{S}^{d-1})}.$

5.3. Bessel functions and solution of the transmission problem

5.3.1. The two-dimensional case. — Given a function $f \in C^0(\mathbb{R}^2; \mathbb{C})$, its restriction to the circle $|x| = R$ can be written as

$$f_R(\theta) := f\big(R(\cos\theta, \sin\theta)\big) = \sum_{n \in \mathbb{Z}} c_n(R) e^{in\theta},$$

where $(|x|, \theta)$ are the polar coordinates centred at the origin. Thanks to Plancherel's identity,

$$\big\| f(|x| = R) \big\|^2_{L^2(\mathbb{S}^1)} = \|f_R\|^2_{L^2(\mathbb{S}^1)} = \int_{\mathbb{S}^1} \big| f(R(\cos\theta, \sin\theta)) \big|^2 d\sigma(\theta) = \sum_{n=-\infty}^{\infty} \big| c_n(R) \big|^2.$$

For $r \in [0, \infty)$ and $n \in \mathbb{Z}$, we write $J_n(r)$ for the non singular Bessel function of order n: that is, the bounded solution of

$$(5.6) \qquad r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}y}{\mathrm{d}r}\right) + (r^2 - n^2)y = 0,$$

normalised near $r = 0$ by

$$J_n(r) \sim \left(\frac{r}{2}\right)^n \frac{1}{\Gamma(n+1)}.$$

The solutions of (5.6) that are linearly independent of $J_n(r)$ are proportional to

$$Y_n(r) + \lambda J_n(r),$$

for some $\lambda \in \mathbb{R}$. Near the origin, we have

$$Y_0(r) \sim \frac{2}{\pi} \log r \quad \text{and} \quad Y_n(r) \sim -\frac{1}{\pi\Gamma(n)}\left(\frac{2}{r}\right)^n.$$

The Hankel function of order n is

$$H_n^{(1)}(r) = J_n(r) + iY_n(r).$$

It is the only normalised solution of (5.6) such that $x \mapsto H_n^{(1)}(|x|)\exp(in\arg(x))$ satisfies the Sommerfeld radiation condition (5.2). The function $H_n^{(1)}(r)$ admits a Laurent series expansion, and can be extended to all $n \in \mathbb{C}$ and $r \in \mathbb{C} \setminus \{0\}$. We will only use it for real arguments.

Proposition 5.7 (see [122]). — *Given $f \in L^2(\mathbb{S}^1)$, $\omega > 0$ and $R > 0$ such that $J_n(\omega R) \neq 0$ for all $n \in \mathbb{Z}$, there exists a unique solution $u \in C^2(B(0, R))$ of*

$$\Delta u + \omega^2 u = 0 \quad \text{in } B(0, R),$$

such that

$$\lim_{|x| \to R} \|u(|x|, \bullet) - f\|_{L^2(\mathbb{S}^1)} = 0$$

and it is given by

$$u(x) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{J_n(\omega|x|)}{J_n(\omega R)} e^{in \arg(x)},$$

where α_n is the n-th Fourier coefficient of f. The series converges uniformly on compact subsets of $B(0, R)$.

Note that $J_{-n}(r) = (-1)^n J_n(r)$ and $Y_{-n}(r) = (-1)^n Y_n(r)$, and for $n \geq 0$ the function $r \mapsto J_n(r)$ is positive on $(0, n]$ **[176]**; in particular, $\omega R < 1$ guarantees that $J_n(\omega R) \neq 0$ for all $n \in \mathbb{Z}$. In the sequel, we will write

$$u(x) = \sum_{n=-\infty}^{\infty} c_n J_n(\omega|x|) e^{in \arg(x)},$$

with suitable coefficients c_n. Since Bessel functions of the first kind are bounded in n and r,

$$\left| J_n(r) \right| < \left(\frac{2}{1 + n^2 + r^2} \right)^{\frac{1}{6}},$$

(see Landau **[132]**) this series is uniformly convergent on compact subsets of $\mathbb{R}^2$ when $\sum |c_n| |n|^{-\frac{1}{3}} < \infty$.

Proposition 5.8 (see [122]). — *Given $\omega > 0$, $R > 0$ and $f \in L^2(\mathbb{S}^1)$, there exists a unique radiating (or scattering) solution $v \in C^2\big(\mathbb{R}^2 \setminus \overline{B(0, R)}\big)$ of*

$$\Delta v + \omega^2 v = 0 \quad \text{in } \mathbb{R}^2 \setminus B(0, R),$$

satisfying (5.2) such that

$$\lim_{|x| \to R} \left\| v(|x|, \bullet) - f \right\|_{L^2(\mathbb{S}^1)} = 0.$$

Furthermore, v admits an expansion of the form

$$v(x) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{H_n^{(1)}(\omega|x|)}{H_n^{(1)}(\omega R)} e^{in \arg(x)},$$

where α_n is the n-th Fourier coefficient of f. The series converges uniformly on compact subsets of $\mathbb{R}^2 \setminus \overline{B(0, R)}$.

5.3.2. The three dimensional case. — The solutions of the radial equation arising from a separation of variables in the Helmholtz equation are the spherical Bessel functions, given by

$$j_n(r) = \sqrt{\frac{2}{\pi r}}\, J_{n+\frac{1}{2}}(r), \quad y_n(r) = \sqrt{\frac{2}{\pi r}}\, Y_{n+\frac{1}{2}}(r), \quad h_n^{(1)}(r) = j_n(r) + \mathrm{i}\, y_n(r).$$

With the same notation used in the two-dimensional case, we denote by f_R the function defined on $\mathbb{S}^2$ by $f_R(\xi) = f(R\xi)$.

Proposition 5.9 (*see* [**122**]). — *Given $f \in L^2(\mathbb{S}^2)$, $\omega > 0$ and $R > 0$ such that $j_n(\omega R) \neq 0$ for all $n \in \mathbb{Z}$, there exists a unique solution $u \in C^2(B(0,R))$ of*

$$\Delta u + \omega^2 u = 0 \quad in \ B(0,R),$$

such that

$$\lim_{|x| \to R} \left\| u(|x|, \bullet) - f \right\|_{L^2(\mathbb{S}^2)} = 0.$$

It is given by

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m} \frac{j_n(\omega|x|)}{j_n(\omega R)} Y_n^m\left(\frac{x}{|x|}\right),$$

where $\alpha_{n,m} = (f, Y_n^m)_{L^2(\mathbb{S}^2)}$ are the coefficients of f relative to its expansion in spherical harmonics.

Similarly, for radiating solutions, we have the following result.

Proposition 5.10 (*see* [**122**]). — *Given $\omega > 0$, $R > 0$, and $f \in L^2(\mathbb{S}^2)$, there exists a unique solution $u \in C^2\big(\mathbb{R}^3 \setminus \overline{B(0,R)}\big)$ of*

$$\Delta v + \omega^2 v = 0 \quad in \ \mathbb{R}^3 \setminus \overline{B(0,R)},$$

satisfying (5.2) such that

$$\lim_{|x| \to R} \left\| v(|x|, \bullet) - f \right\|_{L^2(\mathbb{S}^2)} = 0.$$

It is given by

$$v = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m} \frac{h_n^{(1)}(\omega|x|)}{h_n^{(1)}(\omega R)} Y_n^m\left(\frac{x}{|x|}\right),$$

where $\alpha_{n,m} = (f, Y_n^m)_{L^2(\mathbb{S}^2)}$. Moreover, this series converges uniformly on compact subsets of $\mathbb{R}^3 \setminus \overline{B(0,R)}$.

5.3.3. Bessel representation formula for the solution of (5.1). **—** We look for a solution of problem (5.1) by writing

$$u(x) = u^t(x)\big(1 - H(|x| - \varepsilon)\big) + \big(u^s(x) + u^i(x)\big)H\big(|x| - \varepsilon\big),$$

where H is the Heaviside function, which is nought on $(-\infty, 0)$ and equal to one on $[0, \infty)$. Since u^t and u^i are non singular, they admit an expansion of the form given by Proposition 5.7 when $d = 2$ (resp. Proposition 5.9 when $d = 3$) whereas u^s is radiating and admits an expansion given by Proposition 5.8 when $d = 2$ (resp. Proposition 5.10 when $d = 3$). Since u and $\nabla u \cdot x$ are continuous at $|x| = \varepsilon$, writing

$$u^i(x) = \sum_{n \in \mathbb{Z}} c_n J_n(\omega|x|) e^{in \arg(x)},$$

$$\Big(\text{resp. } u^i(x) = \sum_{n \in \mathbb{N}} \sum_{m=-n}^{n} c_{n,m} j_n(\omega|x|) Y_n^m\Big(\frac{x}{|x|}\Big)\Big)$$

we obtain that the scattered field u^s is given by

(5.7)
$$u^s(x) = \sum_{n \in \mathbb{Z}} R_n(\omega\varepsilon, a) c_n H_n^{(1)}(\omega|x|) e^{in \arg(x)}$$

(5.8)
$$\Big(\text{resp. } u^s(x) = \sum_{n \in \mathbb{N}} \sum_{m=-n}^{n} r_n(\omega\varepsilon, a) c_{n,m} h_n^{(1)}(\omega|x|) Y_n^m\Big(\frac{x}{|x|}\Big)\Big)$$

with $R_n : (0, \infty) \times (0, \infty) \longrightarrow \mathbb{C}$,

$$(r, \lambda) \longmapsto -\frac{\mathrm{Re}\big(H_n^{(1)\prime}(r) J_n(r\lambda) - \lambda J_n'(\lambda r) H_n^{(1)}(r)\big)}{H_n^{(1)\prime}(r) J_n(r\lambda) - \lambda J_n'(\lambda r) H_n^{(1)}(r)},$$

and $r_n = R_{n+\frac{1}{2}}$.

Notation. — Given $n \in \mathbb{Z}$, we use the notation

$$\nu_n = |n| + \tfrac{1}{2}(d - 2),$$

where $d = 2$ or 3 is the dimension of the ambient space. Given $S \subseteq \mathbb{N}$, we note

$$\pm S = \big\{n \in \mathbb{Z} : \exists p \in S \text{ s.t. } n = p \text{ or } n = -p\big\}.$$

Lemma 5.11 (see [75]). — *Given* $\nu \in [1, \infty)$, $\lambda > 0$, *and* $r \leq \min(\nu, \frac{\nu}{\lambda})$, *there holds*

$$\frac{|R_\nu(r, \lambda) H_\nu^{(1)}(r)|}{|J_\nu(r)|} < \frac{2r}{\nu^{1/3}} |\lambda - 1|.$$

Lemma 5.11 is the last preliminary result used to proved Theorem 5.1.

Proof of Theorem 5.1. — Our approach is to choose a representation of the incident field which makes the quantity

$$\left\| u^i(|x| = \varepsilon) \right\|_{H_*^s(\mathbb{S}^{d-1})}$$

appear more readily. Let us suppose $d = 2$ first. Under the form

$$u^i(x) = \sum_{n \in \mathbb{Z}} c_n J_n(\omega |x|) \, e^{i n \arg(x)},$$

the norm of interest does not have a simple expression, in terms of the sequence c_n. On the other hand, if we could write

$$u^i(x) = \sum_{n \in \mathbb{Z}} \alpha_n \frac{J_n(\omega |x|)}{J_n(\omega \varepsilon)} e^{i n \arg(x)},$$

then we would have

$$\left\| u^i(|x| = \varepsilon) \right\|_{H_*^s(\mathbb{S}^{d-1})}^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} (1 + n^2)^s |\alpha_n|^2.$$

We therefore divide the indices into two categories: *the favourable ones,*

$$N_\varepsilon = \left\{ n \in \mathbb{N} \setminus \{0\} : \omega \max(a, 1) \varepsilon \leq j'_{\nu_n, 1} \right\},$$

where $j'_{\nu, 1} \in (\nu, \infty)$ is the first positive zero of $r \to J'_\nu(r)$; and the others,

$$M_\varepsilon = \mathbb{N} \setminus N_\varepsilon.$$

Note that N_ε is larger than M_ε, which is finite, for any given $|x|$ and ω. It is nevertheless the good set, as when $n \in N_\varepsilon$ then $J_{\nu_n}(\omega \varepsilon) > 0$; we thus can apply Proposition 5.7 and find that

$$u^i(x) = \sum_{n \in \pm N_\varepsilon} \alpha_n \frac{J_n(\omega |x|)}{J_n(\omega \varepsilon)} e^{i n \arg(x)} + \sum_{n \in \pm M_\varepsilon} c_n J_n(\omega |x|) e^{i n \arg(x)}.$$

Following the same approach when $d = 3$ and applying Proposition 5.9 we find that the incident field u^i admits a representation of the form

$$u^i(x) = \sum_{n \in N_\varepsilon} \sum_{m=-n}^{n} \alpha_{n,m} \frac{j_n(\omega|x|)}{j_n(\omega\varepsilon)} Y_n^m\left(\frac{x}{|x|}\right) + \sum_{n \in M_\varepsilon} \sum_{m=-n}^{n} c_{n,m} j_n(\omega|x|) Y_n^m\left(\frac{x}{|x|}\right).$$

This leads to the following norm representation, valid in both dimensions,

$$(5.9) \quad \left\|u^i(|x| = r)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2 = \left(\frac{\varepsilon}{r}\right)^{d-2} \sum_{n \in N_\varepsilon} (1+n^2)^s |\alpha_{\pm n}|^2 \cdot \left|\frac{J_{\nu_n}(\omega r)}{J_{\nu_n}(\omega\varepsilon)}\right|^2$$

$$+ \sum_{n \in M_\varepsilon \setminus \{0\}} (1+n^2)^s |c_{\pm n}|^2 \left(\frac{2}{\pi\omega r}\right)^{d-2} |J_{\nu_n}(\omega r)|^2,$$

with the notation $|\alpha_{\pm n}|^2 = |\alpha_{-n}|^2 + |\alpha_n|^2$ when $d = 2$ and $|\alpha_n|^2 = \sum_{m=-n}^{n} |\alpha_{n,m}|^2$ when $d = 3$ (similarly for $|c_{\pm n}|^2$ and $|c_n|^2$).

We have assumed that $\omega\varepsilon \max(1, a) \leq 1$. Then $\omega\varepsilon \max(1, a) \leq j'_{\nu_n, 1}$ for all n, and $M_\varepsilon = \{0\}$. Therefore thanks to (5.7) and (5.8), we have for $R \geq \varepsilon$,

$$\left\|u^s(|x| = R)\right\|_{H_*^{s+1/3}(\mathbb{S}^{d-1})}^2 = \left(\frac{\varepsilon}{R}\right)^{d-2} \sum_{n \in N_\varepsilon} (1+n^2)^{s+\frac{1}{3}} |\alpha_{\pm n}|^2 \cdot \left|\frac{R_{\nu_n}(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega R)}{J_{\nu_n}(\omega\varepsilon)}\right|^2.$$

For all $n \in \mathbb{Z} \setminus \{0\}$, $r \mapsto r|H_{\nu_n}^{(1)}(r)|^2$ is a decreasing function (see [**209**, 13.74]); since $\varepsilon \leq R$, this implies in particular

$$(5.10) \qquad \left|R_{\nu_n}(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega R)\right|^2 \leq \frac{\varepsilon}{R} \left|R_{\nu_n}(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega\varepsilon)\right|^2.$$

Thanks to Lemma 5.11, the bound (5.10) shows in turn that

$$\left|\frac{R_{\nu_n}(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega R)}{J_{\nu_n}(\omega\varepsilon)}\right|^2 < \frac{\varepsilon}{R} \left(\frac{2\omega\varepsilon}{\nu_n^{1/3}} |a - 1|\right)^2 \leq \frac{\varepsilon}{R} \frac{4(\omega\varepsilon)^2}{\nu_n^{2/3}} (a - 1)^2,$$

for all $n \in N_\varepsilon$. This implies that

$$\left\|u^s(|x| = R)\right\|_{H_*^{s+1/3}(\mathbb{S}^{d-1})}^2 \leq C^2(\omega\varepsilon)^2 \left(\frac{\varepsilon}{R}\right)^{d-1} (a - 1)^2 \sum_{n \in N_\varepsilon} (1+n^2)^s |\alpha_{\pm n}|^2$$

$$= C^2(\omega\varepsilon)^2 \left(\frac{\varepsilon}{R}\right)^{d-1} (a - 1)^2 \left\|u^i(|x| = \varepsilon)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2,$$

with $C \leq 2 \max_{n \in \mathbb{N}^*} ((1 + n^2)/\nu_n^2)^{\frac{1}{3}} < 3$, which is the announced bound. □

Having in mind that on the ball centred at the origin of radius R we have

$$(5.11) \qquad \int_{B_R} f \, dx = \int_0^R \Big(\int_{\mathbb{S}^{d-1}} f(r\sigma) \, d\sigma \Big) r^{d-1} \, dr,$$

we define, for all $R > 0$

$$\|f\|^2_{L^2(0,R;H^s_*(\mathbb{S}^{d-1}))} := \int_0^R \|f(|x| = r)\|^2_{H^s_*(\mathbb{S}^{d-1})} r^{d-1} \, dr.$$

In order to prove Theorem 5.2, we are going to show a slightly stronger result.

Lemma 5.12. — *For all $R \geq \max(1,a)\varepsilon$, $\omega > 0$ and $s \geq 0$ there holds*

$$\|u^s(|x| = R)\|^2_{H^s_*(\mathbb{S}^{d-1})} \leq 8 \Big(\frac{\varepsilon}{R} \Big)^{d-1} \|u^i(|x| = \varepsilon)\|^2_{H^s_*(\mathbb{S}^{d-1})}$$
$$+ 3\omega^2 \frac{\varepsilon \max(a,1)}{R^{d-1}} \|u^i\|^2_{L^2(0,\max(1,a)\varepsilon;H^{s-2/3}_*(\mathbb{S}^{d-1}))}.$$

Proof of Theorem 5.2. — Remark that

$$\|u^i\|^2_{L^2(0,\max(1,a)\varepsilon;H^{s-2/3}_*(\mathbb{S}^{d-1}))} \leq \|u^i\|^2_{L^2(0,\max(1,a)\varepsilon;H^s_*(\mathbb{S}^{d-1}))}$$
$$= \int_0^{\max(1,a)\varepsilon} \|u^i(|x| = r)\|^2_{H^s_*(\mathbb{S}^{d-1})} r^{d-1} \, dr$$
$$\leq \sup_{0 \leq r \leq \max(1,a)\varepsilon} \|u^i(|x| = r)\|^2_{H^s_*(\mathbb{S}^{d-1})} \frac{(\max(1,a)\varepsilon)^d}{d},$$

therefore Lemma 5.12 implies that

$$\|u^s(|x| = R)\|^2_{H^s_*(\mathbb{S}^{d-1})}$$
$$\leq 8 \Big(\frac{\max(a,1)\varepsilon}{R} \Big)^{d-1} \sup_{0 \leq r \leq \max(1,a)\varepsilon} \|u^i(|x| = r)\|^2_{H^s_*(\mathbb{S}^{d-1})} \big(1 + \omega^2 \varepsilon^2 \max(a,1)^2 \big),$$

which establishes our claim. $\qquad \square$

The proof of Lemma 5.12 is based on the following lemma.

Lemma 5.13 (see [75]). — *For all $r > 0$ and $\lambda > 0$ there holds*

$$\big| R_{\nu_n}(r,\lambda) \big| \leq 1.$$

For all $\lambda > 0$ and $r > 0$ such that $0 \leq \max(\lambda, 1) r \leq j'_{\nu_n, 1}$ there holds

$$(5.12) \qquad \frac{\left| R_{\nu_n}(r, \lambda) H^{(1)}_{\nu_n}(r) \right|}{\left| J_{\nu_n}(r) \right|} \leq 2^{\frac{4}{3}},$$

where $j'_{\nu_n, 1} \in (\nu_n, \infty)$ is the first positive zero of $r \to J'_{\nu_n}(r)$.

Proof of Lemma 5.12. — We adopt the notations used in the proof of Theorem 5.1. Namely we write

$$(5.13) \quad \left\| u^i(|x| = r) \right\|^2_{H^s_*(\mathbb{S}^{d-1})} = \left(\frac{\varepsilon}{r} \right)^{d-2} \sum_{n \in N_\varepsilon} (1 + n^2)^s |\alpha_{\pm n}|^2 \cdot \left| \frac{J_{\nu_n}(\omega r)}{J_{\nu_n}(\omega \varepsilon)} \right|^2$$

$$+ \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^s |c_{\pm n}|^2 \left(\frac{2}{\pi \omega r} \right)^{d-2} \left| J_{\nu_n}(\omega r) \right|^2,$$

where $N_\varepsilon = \{ n \in \mathbb{N} \setminus \{0\} : \omega \max(a, 1) \varepsilon \leq j'_{\nu_n, 1} \}$ and $M_\varepsilon = \mathbb{N} \setminus N_\varepsilon$. Thanks to (5.7) and (5.8), we have for $R \geq \varepsilon$,

$$\left\| u^s(|x| = R) \right\|^2_{H^s_*(\mathbb{S}^{d-1})}$$

$$= \sum_{n \in N_\varepsilon} (1 + n^2)^s |\alpha_{\pm n}|^2 \cdot \left| \frac{R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega R)}{J_{\nu_n}(\omega \varepsilon)} \right|^2 \left(\frac{\varepsilon}{R} \right)^{d-2},$$

$$+ \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^s |c_{\pm n}|^2 \left(\frac{2}{\pi \omega R} \right)^{d-2} \left| R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega R) \right|^2.$$

For all $n \in \mathbb{Z} \setminus \{0\}$, $r \mapsto r |H^{(1)}_{\nu_n}(r)|^2$ is a decreasing function; thus, using $\varepsilon \leq R$ we have

$$\left| R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega R) \right|^2 \leq \frac{\varepsilon}{R} \cdot \left| R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega \varepsilon) \right|^2, \quad n \in N_\varepsilon,$$

and using $\max(a, 1) \varepsilon \leq R$ we have for $n \in M_\varepsilon \setminus \{0\}$

$$\left| R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega R) \right|^2 \leq \frac{\varepsilon \max(a, 1)}{R} \cdot \left| R_{\nu_n}(\omega \varepsilon, a) H^{(1)}_{\nu_n}(\omega \max(a, 1) \varepsilon) \right|^2,$$

which yield

$$
\left\| u^s(|x| = R) \right\|_{H_*^s(\mathbb{S}^{d-1})}^2
$$

$$
\leq \left(\frac{\varepsilon}{R}\right)^{d-1} \sum_{n \in N_\varepsilon} (1 + n^2)^s |\alpha_{\pm n}|^2 \cdot \left| \frac{R_n(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega\varepsilon)}{J_{\nu_n}(\omega\varepsilon)} \right|^2
$$

$$
+ \frac{\varepsilon \max(a, 1)}{R^{d-1}} \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^s |c_{\pm n}|^2 \left(\frac{2}{\pi\omega}\right)^{d-2} \left| H_{\nu_n}^{(1)}(\omega \max(a, 1)\varepsilon) \right|^2,
$$

where in the second sum we used that $|R_{\nu_n}(\omega\varepsilon, a)| \leq 1$ (Lemma 5.13). Thanks to Lemma 5.13 we have

$$
(5.14) \qquad \sum_{n \in N_\varepsilon} (1 + n^2)^s |\alpha_{\pm n}|^2 \cdot \left| \frac{R_{\nu_n}(\omega\varepsilon, a) H_{\nu_n}^{(1)}(\omega\varepsilon)}{J_{\nu_n}(\omega\varepsilon)} \right|^2
$$

$$
\leq 2^{\frac{8}{3}} \sum_{n \in N_\varepsilon} (1 + n^2)^s |\alpha_{\pm n}|^2 \leq 8 \left\| u^i(|x| = \varepsilon) \right\|^2.
$$

On the other hand, Lemma 5.14 below shows that for all $n \in M_\varepsilon$

$$
\left| H_{\nu_n}^{(1)}(\omega \max(a, 1)\varepsilon) \right|^2 \leq 3\omega^2 (1 + n^2)^{-\frac{2}{3}} \int_0^{\max(1,a)\varepsilon} \left| J_{\nu_n}(\omega r) \right|^2 r \, dr.
$$

As a consequence, we have

$$
(5.15) \qquad \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^s |c_{\pm n}|^2 \left(\frac{2}{\pi\omega}\right)^{d-2} \left| H_{\nu_n}^{(1)}(\omega \max(a, 1)\varepsilon) \right|^2
$$

$$
\leq 3\omega^2 \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^{s - \frac{2}{3}} |c_{\pm n}|^2 \left(\frac{2}{\pi\omega}\right)^{d-2} \int_0^{\max(1,a)\varepsilon} \left| J_{\nu_n}(\omega r) \right|^2 r \, dr
$$

$$
= 3\omega^2 \int_0^{\max(1,a)\varepsilon} \sum_{n \in M_\varepsilon \setminus \{0\}} (1 + n^2)^{s - \frac{2}{3}} |c_{\pm n}|^2 \left(\frac{2}{\pi\omega r}\right)^{d-2} \left| J_{\nu_n}(\omega r) \right|^2 r^{d-1} \, dr
$$

$$
\leq 3\omega^2 \| u^i \|_{L^2(0, \max(1,a)\varepsilon; H_*^{s-2/3}(\mathbb{S}^{d-1}))}^2.
$$

Combining (5.14) and (5.15) we have obtained

$$
\left\| u^s(|x| = R) \right\|_{H_*^s(\mathbb{S}^{d-1})}^2 \leq 8 \left(\frac{\varepsilon}{R}\right)^{d-1} \left\| u^i(|x| = \varepsilon) \right\|_{H_*^s(\mathbb{S}^{d-1})}^2
$$

$$
+ 3\omega^2 \cdot \frac{\varepsilon \max(a, 1)}{R^{d-1}} \| u^i \|_{L^2(0, \max(1,a)\varepsilon; H_*^{s-2/3}(\mathbb{S}^{d-1}))}^2,
$$

as announced. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Lemma 5.14. — *For any $\tau > 0$, provided that $\omega\tau \geq j'_{\nu_n,1}$, we have*

$$\left|H^{(1)}_{\nu_n}(\omega\tau)\right|^2 \leq 3\omega^2(1+n^2)^{-\frac{2}{3}} \int_0^\tau \left|J_{\nu_n}(\omega r)\right|^2 r\,dr.$$

Proof. — We compute that

$$(5.16) \quad \omega^2 \int_0^\tau \left|J_{\nu_n}(\omega r)\right|^2 r\,dr \geq \int_0^{j'_{\nu_n,1}} \left|J_{\nu_n}(r)\right|^2 r\,dr = \frac{(j'_{\nu_n,1})^2 - \nu_n^2}{2}\left|J_{\nu_n}(j'_{\nu_n,1})\right|^2,$$

where the last identity follows from the differential equation satisfied by Bessel functions. More precisely, given $s > 0$, as $w_s = \sqrt{r}\,J_{\nu_n}(sr)$ satisfies

$$-w''_s(r) + \frac{\nu_n^2 - \frac{1}{4}}{r^2} w_s(r) = s^2 w_s(r),$$

we find $(s^2 - 1)w_1 w_s = w''_1 w_s - w''_s w_1$. Thus, by an integration by parts we have that for $\sigma > 0$

$$\int_0^\sigma w_1^2\,dr = \lim_{s\to 1} \int_0^\sigma w_1 w_s\,dr$$

$$= \lim_{s\to 1}(s^2 - 1)^{-1}(w'_1 w_s - w'_s w_1)\big|_0^\sigma$$

$$= \lim_{s\to 1}(s^2 - 1)^{-1}\left(w'_1(\sigma)w_s(\sigma) - w'_s(\sigma)w_1(\sigma)\right).$$

For $\sigma = j'_{\nu_n,1}$, since $J'_{\nu_n}(j'_{\nu_n,1}) = 0$, this becomes

$$\int_0^{j'_{\nu_n,1}} \left|J_{\nu_n}(r)\right|^2 r\,dr = \lim_{s\to 1} \frac{sj'_{\nu_n,1}}{1 - s^2} J_{\nu_n}(j'_{\nu_n,1}) J'_{\nu_n}(sj'_{\nu_n,1}).$$

Taking the limit as $s \to 1$, by (5.6) we obtain (5.16). When $r \geq \nu_n$, there holds $\pi\sqrt{r^2 - \nu_n^2}\,|H^{(1)}_{\nu_n}(r)|^2 \leq 2$ (see [**209**, 13.74]). As a consequence, as $j'_{\nu_n,1} > \nu_n$, we have

$$(5.17) \qquad\qquad \left|H^{(1)}_{\nu_n}(\omega\tau)\right|^2 \leq \frac{2}{\pi}\left((j'_{\nu_n,1})^2 - \nu_n^2\right)^{-\frac{1}{2}}.$$

Combining (5.16) and (5.17) we find

$$\left|H^{(1)}_{\nu_n}(\omega\tau)\right|^2 \leq \omega^2 C_{\nu_n} \int_0^\tau \left|J_{\nu_n}(\omega r)\right|^2 r\,dr,$$

with $C_\nu = \frac{4}{\pi}\big((j'_{\nu,1})^2 - \nu^2\big)^{-\frac{3}{2}}|J_\nu(j'_{\nu,1})|^{-2}$. The final estimate for the constant C_{ν_n} follows from known bounds on zeros of Bessel functions. For all $\nu \geq 1$,

$$\big|J_\nu(j'_{\nu,1})\big| > C_1 \nu^{-\frac{1}{3}} \qquad \text{(see [132])},$$

$$j'_{\nu,1} > \nu + C_2 \nu^{\frac{1}{3}} \qquad \text{(see [176, 10.21.40])}$$

with explicit positive constants C_1 and C_2. This yields in turn the upper bound

$$\frac{4}{\pi}\big((j'_{\nu_n,1})^2 - \nu_n^2\big)^{-\frac{3}{2}}\big|J_{\nu_n}(j'_{\nu_n,1})\big|^{-2} < 3(1 + n^2)^{-\frac{2}{3}},$$

which concludes our argument. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us now turn to lower bound estimates, in the case of a contrast greater than one. This phenomenon is due to the different behaviour of $|J_{\nu_n}(\nu_n r)|$, corresponding to the incident field, and $|H^{(1)}_{\nu_n}(\nu_n r)|$, corresponding to the scattered field, for $r < 1$.

Lemma 5.15 (see [75]). — *For all $\nu \geq 1$ and $r < 1$ there holds*

$$\big|J_\nu(\nu r)\big| \leq \frac{1}{2\nu^{\frac{1}{3}}}\exp\big(-\nu(1-r)^{\frac{3}{2}}\big),$$

$$\big|H^{(1)}_\nu(\nu r)\big| \geq \sqrt{\tfrac{3}{5}} \cdot \frac{1}{\nu^{\frac{1}{3}}}\exp\big(\tfrac{2}{5}\nu(1-r)^{\frac{3}{2}}\big).$$

Proof. — It is shown in [75, Appendix A] that

$$g := r \mapsto \log J_\nu(\nu r) \quad \text{and} \quad k := r \mapsto -\log|Y_\nu(\nu r)|$$

satisfy, for all $0 < r < 1$,

$$g'(r) \geq \nu\sqrt{\frac{1}{r^2} - 1} \quad \text{and} \quad k'(r) \geq \frac{2}{5}g'(r).$$

Note that $\int_r^1 \sqrt{1/t^2 - 1}\,dt > (1-r)^{\frac{3}{2}}$ for $0 < r < 1$, and therefore

$$J_\nu(\nu r) < J_\nu(\nu)\exp(-\nu(1-r)^{\frac{3}{2}}) < \frac{1}{2\nu^{\frac{1}{3}}}\exp(-\nu(1-r)^{\frac{3}{2}}),$$

$$\big|H^{(1)}_\nu(\nu r)\big| \geq |Y_\nu(\nu r)| > |Y_\nu(\nu)|\exp\big(\tfrac{2}{5}\nu(1-r)^{\frac{3}{2}}\big) > \sqrt{\tfrac{3}{5}} \cdot \frac{1}{\nu^{\frac{1}{3}}}\exp\big(\tfrac{2}{5}\nu(1-r)^{\frac{3}{2}}\big),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The previous lemma shows that if, for a frequency less than n, an incident wave of the amplitude $J_{\nu_n}(\omega r)$ is scattered with a reflection amplitude of order 1, the scattered field will be extremely large compared to the incident field, almost like a resonance phenomenon: we can call such frequencies *quasi-resonances*. Quasi-resonances exist, as explained by the following lemma.

Lemma 5.16 (see [75]). — *For any $\lambda > 1$ and any $\tau > 1$, there exists $N_0 \in \mathbb{N}^*$ such that for all $n \geq N_0$, there exists $r = r(n, \lambda, \tau) \in (\nu_n/\lambda, \tau/\lambda\nu_n)$ such that*

$$(5.18) \qquad \mathrm{Im}\left(H_n^{(1)\prime}(r) J_n(r\lambda) - \lambda J_n'(\lambda r) H_n^{(1)}(r)\right) = 0,$$

and therefore

$$R_{\nu_n}(r(n, \lambda, \tau), \lambda) = -1.$$

We are now in a position to prove Theorem 5.4.

Proof of Theorem 5.4. — We consider only the two dimensional case, the three dimensional case is similar. For an incident wave $u_i(x) = \exp(i\omega x \cdot \zeta)$, with $|\zeta| = 1$, we have from the Jacobi–Anger expansion, writing $x \cdot \zeta = |x| \cos\theta$

$$u_i(x) = J_0(\omega|x|) + 2\sum_{n=1}^{\infty} i^n J_n(\omega|x|) \cos(n\theta),$$

thus

$$\left\|u^s(|x| = R)\right\|_{H_*^s(\mathbb{S}^1)}^2 = 4\sum_{n>0}(1 + n^2)^s \left|R_n(\omega\varepsilon, a) H_n^{(1)}(\omega R)\right|^2.$$

Given $\varepsilon \leq R < a\varepsilon$, let $\tau = \sqrt{a\varepsilon/R}$, $\lambda = a$ and $n > N_0$ where N_0 is defined in Lemma 5.16. Choose

$$\omega_n = \varepsilon^{-1} r(n, a, \tau) \in \left(\frac{n}{a\varepsilon}, \frac{n}{a\varepsilon}\tau\right),$$

where r is defined in Lemma 5.16. Then,

$$(5.19) \qquad \left\|u_n^s(|x| = R)\right\|_{H_*^s(\mathbb{S}^1)} \geq 2(1 + n^2)^{\frac{1}{2}s}\left|H_n^{(1)}(\omega_n R)\right|.$$

As $\omega_n R < n\tau^{-1} < n$, Lemma 5.15 shows that

$$(5.20) \qquad \left|H_n^{(1)}(\omega_n R)\right| > \exp\left(\tfrac{2}{5}n\left(1 - \sqrt{R/a\varepsilon}\right)^{\frac{3}{2}}\right)\sqrt{\tfrac{3}{5}}\,n^{-\frac{1}{3}}.$$

Combining (5.19) and (5.20) we obtain

$$\left\|u_n^s(|x| = R)\right\|_{H_*^s(\mathbb{S}^1)} > C \exp\left(\tfrac{2}{5}n\left(1 - \sqrt{R/a\varepsilon}\,\right)^{\frac{3}{2}}\right)\sqrt{1 + n^2}^{\,s - \frac{1}{3}},$$

for some absolute constant $C > 0$, as announced. To conclude the proof, note that while (5.18) has only one solution in the given range, $x \mapsto R_{\nu_n}(x, a)$ is continuous, therefore there is an open interval (ω_n^-, ω_n^+) contained in $\left(\frac{n}{a\varepsilon}, \frac{n}{a\varepsilon}\tau\right)$ where $|R_{\nu_n}(x, \lambda)| \geq \frac{1}{2}$. Therefore for any $\omega \in (\omega_n^-, \omega_n^+)$ we have

$$\left\|u^s(|x| = R)\right\|_{H_*^s(\mathbb{S}^1)} \geq (1 + n^2)^{\frac{1}{2}s}\left|H_n^{(1)}(\omega R)\right|.$$

The rest of the proof follows, with a final lower bound reduced by $\frac{1}{2}$.   □

We conclude this section by showing how Theorem 5.6 follows from the estimates given in [75], [76] and Theorem 5.1.

Proof of Theorem 5.6. — When $d = 2$, the estimate provided in [75], [76] (using our notation convention) is that if $a < 1$ and $\omega\varepsilon < y_{0,1}$ then

$$\left\|u^s(|x| = R)\right\|_{H^{s+1/3}(\mathbb{S}^{d-1})}^2 \leq \left(3|a - 1|\omega\varepsilon\right)^2\left(\frac{\varepsilon}{R}\left\|u^i(|x| = \varepsilon)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2\right.$$
$$\left. + \frac{1}{2\pi}\left(3|u^i(0)|\omega\varepsilon|H_0^{(1)}(\omega R)|\right)^2\right),$$

whereas when $a > 1$ and $\omega\varepsilon < \min(\frac{1}{2}, \frac{1}{a\sqrt{\log a + 1}})$ then

$$\left\|u^s(|x| = R)\right\|_{H^{s+1/3}(\mathbb{S}^{d-1})}^2 \leq \left(3|a - 1|\omega\varepsilon\right)^2\left(\frac{\varepsilon}{R}\left\|u^i(|x| = \varepsilon)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2\right.$$
$$\left. + \frac{1}{2\pi}\left(8|u^i(0)| \cdot \omega\varepsilon \cdot |H_0^{(1)}(\omega R)|\right)^2\right),$$

where $y_{0,1} \approx 0.89$. As $\sqrt{x}|H_0^{(1)}(x)| \leq \sqrt{2/\pi}$, this implies that when $\omega\varepsilon < \frac{1}{2}$ we have

$$\frac{1}{2\pi}\left(8\omega\varepsilon|H_0^{(1)}(\omega R)|\right)^2 = \frac{\varepsilon}{R} \cdot \frac{8\omega\varepsilon}{\pi^2} < \frac{\varepsilon}{R}J_0(\tfrac{1}{2})^2,$$

and since

$$\left\|u^i(|x| = \varepsilon)\right\|_{H^s(\mathbb{S}^{d-1})}^2 = \left\|u^i(|x| = \varepsilon)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2 + \left|u^i(0)\right|^2 J_0(\omega\varepsilon),$$
$$> \left\|u^i(|x| = \varepsilon)\right\|_{H_*^s(\mathbb{S}^{d-1})}^2 + \left|u^i(0)\right|^2 J_0(\tfrac{1}{2})^2,$$

the result follows.   □

5.4. Appendix on the $H^s_*(\mathbb{S}^{d-1})$ norms used in this chapter

The ambient space is $\mathbb{R}^d$, with $d = 2$ or $d = 3$.

When $d = 2$, given a function $f \in C^0(\mathbb{R}^2)$, its restriction to the circle $|x| = R$ can be written

$$f\big(|x| = R\big) = \sum_{n \in \mathbb{Z}} c_n(R)\, e^{in\arg(x)},$$

where $(|x|, \theta)$ are the polar coordinates centred at the origin. Thanks to Plancherel's identity,

$$\big\| f(|x| = R) \big\|^2_{L^2(\mathbb{S}^1)} = \int_{\mathbb{S}^1} f^2(R\theta)\, d\sigma(\theta) = \sum_{n=-\infty}^{\infty} \big| c_n(R) \big|^2.$$

We write $H^s(\mathbb{S}^1)$ for the Sobolev space of order s on the circle, endowed with the norm

$$\big\| f(|x| = R) \big\|^2_{H^s(\mathbb{S}^1)} = \sum_{n=-\infty}^{\infty} (1 + n^2)^s \big| c_n(R) \big|^2.$$

For simplicity, we will focus on estimates of the non radial component of the scattered field, and will therefore use the norm

$$\big\| f(|x| = R) \big\|_{H^s_*(\mathbb{S}^1)} = \left\| f(|x| = R) - \frac{1}{|\mathbb{S}^1|} \int_{\mathbb{S}^1} f\big(|x| = R\big)\, d\sigma \right\|_{H^s(\mathbb{S}^1)}.$$

When $d = 3$, the above notations have a natural extension using spherical harmonics instead of Fourier coefficients. Given any $f \in C^0(\mathbb{R}^3)$, its restriction to the circle $|x| = R$ can be decomposed in terms of the spherical harmonics (see *e.g.* [176] for definition and properties) in the following way

$$f\left(R\frac{x}{|x|}\right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{n,m}(R) Y_n^m\left(\frac{x}{|x|}\right),$$

where $c_{n,m}(R) = \big(f(|x| = R), Y_n^m\big)_{L^2(\mathbb{S}^2)}$, and

$$\big(f(|x| = R), Y_n^m\big) Y_n^m\left(\frac{x}{|x|}\right) = (2n+1) \int_{\mathbb{S}^2} f(R\,\widehat{y}) P_n\left(\frac{x}{|x|} \cdot \widehat{y}\right) ds(\widehat{y}),$$

where P_n is the n-th Legendre polynomial. The Sobolev spaces $H^s(\mathbb{S}^2)$ norms are given by

$$\big\| f(|x| = R) \big\|^2_{H^s(\mathbb{S}^2)} = \sum_{n=0}^{\infty} (1 + n^2)^s \big| c_n(R) \big|^2,$$

with the notation $|c_n(R)|^2 := \sum_{m=-n}^n |c_{n,m}(R)|^2$, and

$$\big\| f(|x| = R) \big\|_{H_*^s(\mathbb{S}^2)} = \bigg\| f\big(|x| = R\big) - \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} f\big(|x| = R\big) \, \mathrm{d}\sigma \bigg\|_{H^s(\mathbb{S}^2)}.$$

CHAPTER 6

THE JACOBIAN OF SOLUTIONS TO THE CONDUCTIVITY EQUATION

6.1. Introduction

The focus of this chapter is the study of the critical points of the solutions to

(6.1)
$$\begin{cases} -\operatorname{div}(a\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain, for $d \geq 2$. We are interested in finding conditions on φ so that u does not have interior critical points, namely $\nabla u(x) \neq 0$ for all $x \in \Omega$, or, more generally, conditions on d boundary values $\varphi_1, \ldots, \varphi_d$ so that the corresponding solutions $u^1, \ldots, u^d$ to (6.1) satisfy

(6.2)
$$\det \begin{bmatrix} \nabla u^1 & \cdots & \nabla u^d \end{bmatrix}(x) > 0,$$

for every $x \in \Omega$. This constraint is related to the local invertibility of the map

$$(u^1, \ldots, u^d) \colon \Omega \longrightarrow \mathbb{R}^d.$$

The ambient space dimension is the most important parameter. In two dimensions, complex analysis tools can be used to tackle the problem. When $a \equiv 1$, the theory of harmonic mappings [91] provides a positive answer to this question, through the Radó-Kneser-Choquet theorem [184], [123], [81]. In particular, if φ^1 and φ^2 are suitably chosen, then $\det[\nabla u^1 \, \nabla u^2] > 0$ in Ω. A brief discussion of this theorem is given in Section 6.2. The proofs

are omitted, since this result will follow as a corollary of the non-constant coefficient case.

The Radó-Kneser-Choquet theorem was extended to the non-constant coefficient case in dimension two by several authors [15], [16], [14], [19], [54], [20] using PDE methods. Proving positivity of the Jacobian corresponds to the absence of critical points of solutions to (6.1) with suitable boundary values φ. In order to visualise this phenomenon, suppose that u has a critical point in x_0. By the maximum principle, u cannot be a local minimum or maximum, and has to be a saddle point. In other words, u has oscillations around x_0, and these oscillations propagate to the boundary of the domain. Thus, φ needs to have, at least, two alternating minima and maxima. On the contrary, when φ is chosen without such oscillations, *e.g.* $\varphi = x_1$ when Ω is convex, the corresponding solution cannot have interior critical points. A self-contained exposition of this theory for smooth coefficients, namely assuming that a is Hölder continuous, is the content of Section 6.3.

When a is only measurable, the gradient of u is defined almost everywhere, and the Jacobian constraint is not defined pointwise. It is possible to enforce (6.2) almost everywhere in Ω, provided that the boundary conditions are suitably chosen, see [19]. This result is stated and discussed in Section 6.4; the proof is omitted.

The situation is completely different in higher dimensions, where results of this type do not hold [134], [64]. A sketch of the counter-examples is presented in Section 6.5. The presentation follows and extends [77], where the construction given in [64] was used to prove that, for any choice of three boundary values, there exists a conductivity such that the corresponding Jacobian vanishes at some point of the domain. In other words, suitable boundary conditions cannot be determined *a priori* independently of a when the dimension of the ambient space is three or higher.

6.2. The Radó-Kneser-Choquet theorem

This section deals with the two-dimensional case with constant isotropic coefficient, namely $a \equiv 1$. This particular situation is of no interest for imaging

and inverse problems, but is mathematically relevant because it opened the way for the results in the general case. For this reason, we have decided to present the main results but to omit their proofs. For full details we refer the interested reader to [**91**], which we follow in this brief survey.

The problem under consideration is to find conditions on two boundary values φ_1 and φ_2 so that $\det[\nabla u^1 \nabla u^2](x) \neq 0$ for all $x \in \Omega$, where u^i is defined by

$$(6.3) \qquad \begin{cases} \Delta u^i = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

In order to be consistent with the literature, throughout this section we shall use the complex notation, with the identification $\mathbb{C} = \mathbb{R}^2$, $z = x_1 + i x_2$, $\Phi = \varphi_1 + i\varphi_2$ and $f(z) = u^1(x) + i u^2(x)$.

Thanks to the Cauchy-Riemann equations, the quantity we are interested in is nothing else than the Jacobian of the harmonic function $f : \Omega \to \mathbb{C}$, namely

$$(6.4) \qquad J_f(z) = \det\left[\nabla u^1(x) \, \nabla u^2(x)\right].$$

We restrict ourselves to study the case where $\Omega = B(0,1)$ is the disc centred in 0 with radius 1. The main result of this section reads as follows.

Theorem 6.1 (**Radó-Kneser-Choquet theorem**). — *Let $D \subseteq \mathbb{C}$ be a bounded convex domain whose boundary is a Jordan curve ∂D. Let $\Phi : \partial B(0,1) \to \partial D$ be a homeomorphism of $\partial B(0,1)$ onto ∂D and f be defined as*

$$\begin{cases} \Delta f = 0 & \text{in } B(0,1), \\ f = \Phi & \text{on } \partial B(0,1). \end{cases}$$

Then $J_f(z) \neq 0$ for all $z \in B(0,1)$.

Usual proof(s) for this result consist of two steps:

1) If $J_f(z) = 0$ for some $z \in B(0,1)$ then $\alpha_1 \nabla u^1(z) + \alpha_2 \nabla u^2(z) = 0$ for some $\alpha_i \in \mathbb{R}$. Thus, it is helpful to consider the function $u = \alpha_1 u^1 + \alpha_2 u^2$, which is harmonic in $B(0,1)$ with boundary value $\alpha_1\varphi_1 + \alpha_2\varphi_2$ and satisfies $\nabla u(z) = 0$.

2) If u is a real-valued harmonic function in $B(0,1)$ such that $u_{|\partial\Omega}$ takes any value at most twice then u has no critical points in $B(0,1)$, namely $\nabla u(z) \neq 0$ for all $z \in \Omega$.

We shall not detail the derivation of these steps here. We refer the reader to Section 6.3, where similar arguments are used to obtain the corresponding result in the non-constant coefficient case.

It is worth mentioning how this result can be read in terms of harmonic mappings. A univalent (one-to-one) mapping $f\colon \Omega \to \mathbb{C}$ is a harmonic mapping if its real and imaginary parts are harmonic in Ω. Under the assumptions of Theorem 6.1, the function f is locally univalent by the inverse mapping theorem, and the argument principle for harmonic functions [2] gives that f is globally univalent. Therefore, f is a harmonic mapping of $B(0,1)$ onto D.

Conversely, Lewy's theorem asserts that if f is locally univalent in D then $J_f(z) \neq 0$ for all $z \in D$ [140]. The local univalency of f is equivalent to the non-vanishing property of the Jacobian of f.

In the next section we shall generalise Theorem 6.1 and consider the case with an arbitrary anisotropic elliptic tensor $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ in a convex bounded domain $\Omega \subseteq \mathbb{R}^2$.

6.3. The smooth case

Let $\Omega \subseteq \mathbb{R}^2$ be a convex bounded domain with Lipschitz boundary and $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ satisfy the uniform ellipticity condition

$$(6.5) \qquad \Lambda^{-1}|\xi|^2 \leq a\xi \cdot \xi \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^2,$$

and the regularity estimate

$$(6.6) \qquad \|a\|_{C^{0,\alpha}(\overline{\Omega})} \leq \Lambda,$$

for some $\alpha \in (0,1)$ and $\Lambda > 0$. Let us recall the main notation. Given two boundary values $\varphi_1, \varphi_2 \in H^1(\Omega; \mathbb{R})$ let $u^i \in H^1(\Omega; \mathbb{R})$ be the weak solutions to

$$\begin{cases} -\operatorname{div}(a\nabla u^i) = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

As before, we look for suitable boundary conditions (independent of a) such that

$$\det\left[\nabla u^1(x)\ \nabla u^2(x)\right] \neq 0, \quad x \in \Omega.$$

Note that classical elliptic regularity theory [**104**] gives $u^i \in C^1_{\mathrm{loc}}(\Omega; \mathbb{R})$, and so the problem is well-posed.

An answer to this problem for a general choice of the boundary values is given in [**54**]. Roughly speaking, it is sufficient that $\Phi = (\varphi_1, \varphi_2)$ maps $\partial\Omega$ onto the boundary of a convex domain. This condition is clearly a generalisation of the assumption in Theorem 6.1. For brevity, we shall consider here only the simplest choice for Φ, namely the identity mapping. That is, we set

$$\varphi_1 = x_1, \quad \varphi_2 = x_2.$$

The main result reads as follows.

Theorem 6.2. — *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex Lipschitz domain and a in $C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ satisfy (6.5). For $i = 1, 2$ let $u^i \in H^1(\Omega; \mathbb{R})$ be the unique solutions to*

$$\begin{cases} -\operatorname{div}(a\nabla u^i) = 0 & \text{in } \Omega, \\ u^i = x_i & \text{on } \partial\Omega. \end{cases}$$

Then $\det \left[\nabla u^1(x)\ \nabla u^2(x) \right] \neq 0$, $x \in \Omega$.

Remark 6.3. — Once this result is established, it is possible to prove that in fact there holds

$$\det \left[\nabla u^1(x)\ \nabla u^2(x) \right] > 0, \quad x \in \Omega.$$

Indeed, for $t \in [0,1]$ define $a_t = ta + (1-t)I$, where I is the 2×2 identity matrix. Clearly, a_t still satisfies (6.5). Denote

$$\gamma_t = \det \left[\nabla u^1_t\ \nabla u^2_t \right],$$

where u^i_t is given by

$$\begin{cases} -\operatorname{div}(a_t \nabla u^i_t) = 0 & \text{in } \Omega, \\ u^i = x_i & \text{on } \partial\Omega. \end{cases}$$

By Theorem 6.2 there holds

(6.7) $$\gamma_t(x) \neq 0, \quad t \in [0,1], \ x \in \Omega.$$

By standard elliptic regularity (see Lemma 8.5), the map $t \mapsto u_t^i \in C^1(\Omega)$ is continuous, and so the map $t \mapsto \gamma_t(x)$ is continuous for every $x \in \Omega$. Therefore, combining $\gamma_0 = \det[\nabla x_1 \ \nabla x_2] = 1$ with (6.7) yields

$$\det[\nabla u_1^1(x) \ \nabla u_1^2(x)] > 0, \quad x \in \Omega.$$

The claim is proved as $u_1^i = u^i$.

A similar proof can be given by using the Brouwer degree. See [**54**] for details.

Remark 6.4. — In fact, it can be proved that the non-zero constraint above holds up to the boundary [**54**], [**20**], namely

$$\det[\nabla u^1(x) \ \nabla u^2(x)] > 0, \quad x \in \overline{\Omega}.$$

As in the outline of the proof of Theorem 6.1, the main step of the proof of this theorem is the study of the absence of critical points of solutions to the conductivity equation with suitable boundary conditions. Namely, can we find conditions on the boundary value φ such that the corresponding solution u to

$$\begin{cases} -\operatorname{div}(a\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

does not have interior critical points, that is, $\nabla u(x) \neq 0$ for all $x \in \Omega$? The answer is positive, and the conditions are related to the number of oscillations of $\varphi|_{\partial\Omega}$, as we mentioned in §6.1. This theory has an almost independent history, see [**15, 16**] and [**14**] which we follow in this exposition.

The study of this problem is based on a local asymptotic expansion of u in terms of homogeneous polynomials. We state here the version of this result given in [**193**, Theorem 7.4.1], and the reader is referred to [**114**], [**58**], [**69**], [**187**], [**113**] for related results.

Proposition 6.5. — *Let* $\Omega \subseteq \mathbb{R}^2$ *be a bounded Lipschitz domain and let a in* $C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ *satisfy* (6.5). *Let* $u \in C^1_{\mathrm{loc}}(\Omega; \mathbb{R})$ *be a non-constant weak solution to*

$$-\operatorname{div}(a\nabla u) = 0 \quad \text{in } \Omega.$$

For any $x_0 \in \Omega$ there exist $n \in \mathbb{N}$ and $A \in \mathbb{R}^2 \setminus \{0\}$ such that as $x \to x_0$

(i) $u(x) = u(x_0) + \rho^{n+1}\big(\cos((n+1)\theta), \sin((n+1)\theta)\big) \cdot A + o(\rho^{n+1})$,

(ii) $\nabla u(x) = \frac{1}{2}(n+1)\rho^n \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix} A + o(\rho^n)$,

where we have used the notation $x - x_0 = \rho(\cos\theta, \sin\theta)$.

We call n the multiplicity of x_0 as a critical point. In particular, x_0 is a critical point of u if and only if $n \geq 1$. Note that every critical point of u has finite multiplicity (which is also a consequence of the unique continuation property [**193**, Theorem 7.3.1]).

As a corollary, it is possible to prove the discreteness of the interior critical points of u and to study the level set of u near a critical point.

Proposition 6.6. — *Assume that the hypotheses of the previous proposition hold.*

1) *The interior critical points of u are isolated.*

2) *Take x_0 and n as before. In a neighbourhood of x_0 the level line $\{x : u(x) = u(x_0)\}$ is made of $n+1$ arcs intersecting with equal angles at x_0.*

Proof. — 1) By contradiction, suppose that we have $x_\ell, x_0 \in \Omega$ such that $x_\ell \to x_0$ and $\nabla u(x_\ell) = \nabla u(x_0) = 0$. Applying Proposition 6.5, part (ii), in x_0 we obtain for some $n \in \mathbb{N}^*$ and $A \in \mathbb{R}^2 \setminus \{0\}$ as $\ell \to \infty$

$$0 = \nabla u(x_\ell) = \frac{1}{2}(n+1)\rho_\ell^n \begin{bmatrix} \cos(n\theta_\ell) & \sin(n\theta_\ell) \\ -\sin(n\theta_\ell) & \cos(n\theta_\ell) \end{bmatrix} A + o(\rho_\ell^n),$$

where $x_\ell - x_0 = \rho_\ell(\cos\theta_\ell, \sin\theta_\ell)$. Up to a subsequence, assume that $\theta_\ell \to \theta_0$ for some $\theta_0 \in [0, 2\pi]$. Taking the limit as $\ell \to \infty$ yields

$$\begin{bmatrix} \cos(n\theta_0) & \sin(n\theta_0) \\ -\sin(n\theta_0) & \cos(n\theta_0) \end{bmatrix} a = 0,$$

this implies $A = 0$, a contradiction.

2) By Proposition 6.5, part (i), we have as $x \to x_0$

$$u(x) = u(x_0) + \rho^{n+1}\big(\cos((n+1)\theta), \sin((n+1)\theta)\big) \cdot A + o(\rho^{n+1}),$$

where $x - x_0 = \rho(\cos\theta, \sin\theta)$. Observing that the set

$$\left\{ x_0 + \rho(\cos\theta, \sin\theta) : (\cos((n+1)\theta), \sin((n+1)\theta)) \cdot A = 0 \right\}$$

is made of $n + 1$ arcs intersecting with equal angles at x_0 concludes the proof.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We are now ready to study the absence of critical points for solutions to the equation $-\operatorname{div}(a\nabla u) = 0$. Several versions of this result can be found in [**15**], [**16**], [**14**].

Proposition 6.7. — *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded Lipschitz domain and $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ satisfy (6.5). Let $\varphi \in H^1(\Omega; \mathbb{R})$ be such that $\partial\Omega$ can be split into two arcs on which respectively φ is a non-decreasing and non-increasing function of the arc-length parameter. Let $u \in H^1(\Omega; \mathbb{R})$ be the weak solution to*

$$\begin{cases} -\operatorname{div}(a\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

*If u is not constant, then $\nabla u(x) \neq 0$, $x \in \Omega$.*

Proof. — By contradiction, suppose that $x_{(0)} \in \Omega$ is a critical point of u. By Proposition 6.6, part 2), in a neighbourhood U of $x_{(0)}$ the level line $\{x \in \Omega : u(x) = u(x_{(0)})\}$ is made of $n + 1$ arcs intersecting with equal angles at $x_{(0)}$ for some $n \geq 1$. More precisely, By Proposition 6.5, part (i), the set $\{x \in U : u(x) > u(x_{(0)})\}$ is made of $n + 1$ connected components U_ℓ^+, namely

$$\left\{ x \in U : u(x) > u(x_{(0)}) \right\} = \bigcup_{\ell=1}^{n+1} U_\ell^+.$$

Moreover, the components U_ℓ^+ alternate with the corresponding connected components U_ℓ^- of $\{x \in U : u(x) < u(x_{(0)})\}$. Write now $\{x \in \Omega : u(x) > u(x_{(0)})\}$ as the union of its connected components:

$$\left\{ x \in \Omega : u(x) > u(x_{(0)}) \right\} = \bigcup_{j \in J} \Omega_j^+$$

Let $j_1, j_2 \in J$ be such that $U_1^+ \subseteq \Omega_{j_1}^+$ and $U_2^+ \subseteq \Omega_{j_2}^+$. By the maximum principle, the maximum that u attains in $\overline{\Omega_{j_1}^+}$ must be attained at some $x_{(1)} \in \partial\Omega$, and
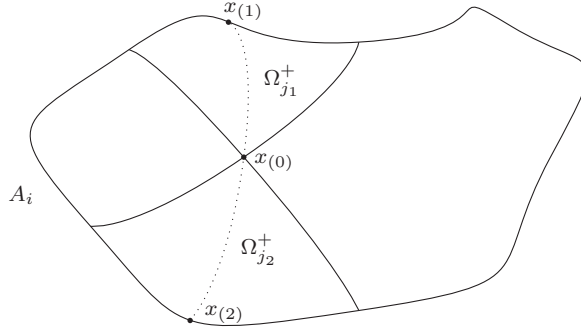
FIGURE 6.1. The level sets of u.

in particular $u(x_{(1)}) > u(x_{(0)})$. Similarly, there exists $x_{(2)} \in \partial\Omega \cap \overline{\Omega^+_{j_2}}$ such that $u(x_{(2)}) > u(x_{(0)})$. See Figure 6.1.

Let us now write $\partial\Omega = A_1 \cup A_2$ as the union of the two arcs starting and terminating at $x_{(1)}$ and $x_{(2)}$. We claim that $A_i \cap \{x \in \partial\Omega : u(x) < u(x_{(0)})\} \neq \varnothing$ for $i = 1, 2$. This contradicts the assumptions on φ, as $u(x_{(1)}) > u(x_{(0)})$ and $u(x_{(2)}) > u(x_{(0)})$, and the proof is concluded.

It remains to show that $A_i \cap \{x \in \partial\Omega : u(x) < u(x_{(0)})\} \neq \varnothing$ for $i = 1, 2$. By contradiction, assume that $A_i \cap \{x \in \partial\Omega : u(x) < u(x_{(0)})\} = \varnothing$ for some $i = 1, 2$. Since $\Omega^+_{j_1}$ is connected by arcs, there exists a path $P_1 \subseteq \overline{\Omega^+_{j_1}}$ connecting $x_{(0)}$ and $x_{(1)}$. Similarly, there exists $P_2 \subseteq \overline{\Omega^+_{j_2}}$ connecting $x_{(0)}$ and $x_{(2)}$. Consider now the domain D surrounded by the boundary $\partial D = A_i \cup P_1 \cup P_2$. By construction, since the components U^+_ℓ and U^-_ℓ alternate around $x_{(0)}$, there exists $\ell = 1, \ldots, n+1$ such that $U^-_\ell \subseteq D$. However, as $u(x) \geq u(x_{(0)})$ for all $x \in \partial D$, by the maximum principle we obtain $u(x) \geq u(x_{(0)})$ for all $x \in D$, a contradiction. $\qquad\square$

We are now ready to prove Theorem 6.2. It remains to study the first step of the outline of the proof of Theorem 6.1, namely to show how to apply Proposition 6.7 to the case of the determinant of the gradients of two different solutions.

Proof of Theorem 6.2. — By contradiction, assume there exists $x_{(0)} \in \Omega$ such that $\det[\nabla u^1(x_{(0)}) \ \nabla u^2(x_{(0)})] = 0$. Thus

$$\alpha_1 \nabla u^1(x_{(0)}) + \alpha_2 \nabla u^2(x_{(0)}) = 0$$

for some $\alpha_i \in \mathbb{R}$ with $\alpha_1^2 + \alpha_2^2 > 0$. Therefore the function $u := \alpha_1 u^1 + \alpha_2 u^2$ satisfies $\nabla u(x_{(0)}) = 0$ and

$$\begin{cases} -\operatorname{div}(a \nabla u) = 0 & \text{in } \Omega, \\ u = \alpha_1 x_1 + \alpha_2 x_2 & \text{on } \partial \Omega. \end{cases}$$

As Ω is convex we claim that the function $(\alpha_1 x_1 + \alpha_2 x_2)|_{\partial \Omega}$ satisfies the assumptions of Proposition 6.7. Indeed, since convexity is preserved by linear transformations, without loss of generality we can assume $\alpha_1 = 1$ and $\alpha_2 = 0$, so that the boundary condition becomes $\varphi(x) = x_1$. Let $x_{(1)}, x_{(2)} \in \partial \Omega$ be such that $\varphi(x_{(1)}) \leq \varphi(x) \leq \varphi(x_{(2)})$ for every $x \in \partial \Omega$. Since Ω is convex, φ is respectively a non-decreasing and non-increasing function of the arc-length parameter on the two arcs connecting $x_{(1)}$ and $x_{(2)}$, and the claim is proved. Further, note that Ω is simply connected. By Proposition 6.7, we obtain $\nabla u(x_{(0)}) \neq 0$, a contradiction. $\qquad \square$

Theorem 6.2 shows how to construct solutions to the conductivity equation satisfying the constraint

$$\left| \det[\nabla u^1(x) \ \nabla u^2(x)] \right| > 0, \quad x \in \Omega.$$

For the applications to hybrid inverse problems, it will be useful to have a quantitative version of this result. This is the content of the following corollary (see [**20**] for a global result).

Corollary 6.8. — *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex Lipschitz domain, $\Omega' \Subset \Omega$ and $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ satisfy (6.5) and (6.6). For $i = 1, 2$ let $u^i \in H^1(\Omega; \mathbb{R})$ be defined as the unique solutions to*

$$\begin{cases} -\operatorname{div}(a \nabla u^i) = 0 & \text{in } \Omega, \\ u^i = x_i & \text{on } \partial \Omega. \end{cases}$$

Then $\left| \det[\nabla u^1(x) \ \nabla u^2(x)] \right| \geq C$ for all $x \in \overline{\Omega'}$, for some $C > 0$ depending only on Ω, Ω', Λ and α.

Note that in view of Remark 6.3, the absolute value in the above inequality can be omitted.

Remark 6.9. — Under the assumption $a \in C^{0,1}$, it is possible to give an explicit expression for the constant C [**17**, Remark 3].

Proof. — We argue by contradiction. If such a constant C did not exist, we would be able to find a sequence $a_n \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times2})$ of tensors satisfying (6.5) and (6.6) such that the corresponding solutions u_n^i to

$$\begin{cases} -\operatorname{div}(a_n \nabla u_n^i) = 0 & \text{in } \Omega, \\ u_n^i = x_i & \text{on } \partial\Omega \end{cases}$$

verify

(6.8) $$\min_{\overline{\Omega'}} \left| \det[\nabla u_n^1 \, \nabla u_n^2] \right| \xrightarrow[n]{} 0.$$

Let $x_n \in \overline{\Omega'}$ be a point where such a minimum is attained. Up to a subsequence, we can suppose that $x_n \to \widetilde{x}$ for some $\widetilde{x} \in \overline{\Omega'} \subseteq \Omega$. By the Ascoli–Arzelà theorem, the embedding $C^{0,\alpha} \hookrightarrow C^{0,\alpha/2}$ is compact. Thus, up to a subsequence, we have that $a_n \to \widetilde{a}$ in $C^{0,\alpha/2}(\overline{\Omega}; \mathbb{R}^{2\times2})$ for some $\widetilde{a} \in C^{0,\alpha/2}(\overline{\Omega}; \mathbb{R}^{2\times2})$ satisfying (6.5) and $\|\widetilde{a}\|_{C^{0,\alpha/2}(\overline{\Omega}; \mathbb{R}^{2\times2})} \leq C(\Omega)\Lambda$.

Let $\widetilde{u}^i$ be the unique solution to

$$\begin{cases} -\operatorname{div}(\widetilde{a} \, \nabla \widetilde{u}^i) = 0 & \text{in } \Omega, \\ \widetilde{u} = x_i & \text{on } \partial\Omega. \end{cases}$$

By looking at the equation satisfied by $u_n^i - \widetilde{u}^i$ and standard elliptic regularity theory [**104**] we find that $\|u_n^i - \widetilde{u}^i\|_{C^1(\overline{\Omega})} \to 0$. Hence

$$\left\| \det[\nabla u_n^1 \, \nabla u_n^2] - \det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2] \right\|_{C^0(\overline{\Omega})} \to 0.$$

Then, assumption (6.8) implies

$$\begin{aligned} |\det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2](\widetilde{x})| \leq \; & |\det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2](\widetilde{x}) - \det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2](x_n)| \\ & + |\det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2](x_n) - \det[\nabla u_n^1 \, \nabla u_n^2](x_n)| \\ & + |\det[\nabla u_n^1 \, \nabla u_n^2](x_n)| \to 0, \end{aligned}$$

which shows that $|\det[\nabla \widetilde{u}^1 \, \nabla \widetilde{u}^2](\widetilde{x})| = 0$ in contradiction to Theorem 6.2. $\square$

6.4. The general case

When the coefficient a of the equation $-\operatorname{div}(a\nabla u^i) = 0$ is not assumed to be Hölder continuous, the problem becomes more complicated. Indeed, u^i may not be continuously differentiable, and so the gradients of solutions may not have meaningful pointwise values. Therefore, the inequality

$$\det\left[\nabla u^1(x)\ \nabla u^2(x)\right] > 0$$

has to be studied almost everywhere in Ω. The first natural attempt is approximating the irregular a with smooth tensors and using Theorem 6.2. Unfortunately, taking the limit transforms the strong inequality above into a weak inequality almost everywhere.

Proposition 6.10. — *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex Lipschitz domain and a in $L^\infty(\Omega;\mathbb{R}^{2\times2})$ satisfy (6.5). For $i = 1,2$ let $u^i \in H^1(\Omega;\mathbb{R})$ be defined as the unique weak solutions to*

$$\begin{cases} -\operatorname{div}(a\nabla u^i) = 0 & \text{in } \Omega, \\ u^i = x_i & \text{on } \partial\Omega. \end{cases}$$

Then $\det[\nabla u^1\ \nabla u^2] \geq 0$ almost everywhere in Ω.

Proof. — The proof is based upon a standard regularisation argument, and only a sketch will be provided.

Using mollifiers, it is possible to show that there exist $a_n \in C^\infty(\overline{\Omega};\mathbb{R}^{2\times2})$ satisfying (6.5) such that $a_n \to a$ in L^p for every $p < \infty$ and almost everywhere. Thanks to Meyers' theorem [157] – see Chapter 3 and Theorem 3.14 – we see that the corresponding solutions u_n^i of

$$\begin{cases} -\operatorname{div}(a_n\nabla u_n^i) = 0 & \text{in } \Omega, \\ u_n^i = x_i & \text{on } \partial\Omega, \end{cases}$$

converge to u^i in $W^{1,2}(\Omega;\mathbb{R})$. Hence, up to a subsequence, ∇u_n^i converges to ∇u^i almost everywhere. As a consequence, the result follows taking the limit in

$$\det\left[\nabla u_n^1(x)\ \nabla u_n^2(x)\right] > 0 \quad x \in \Omega,$$

which is a consequence of Theorem 6.2 and Remark 6.3.   □

Unfortunately, this result is of no practical use in the applications we have in mind, where non-zero quantities are needed. A much better result would be an inequality of the type

$$\det[\nabla u^1 \ \nabla u^2] > 0 \quad \text{almost everywhere in } \Omega.$$

This result was proved in [**19**].

Theorem 6.11. — *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded convex Lipschitz domain, $\Omega' \Subset \Omega$ and $a \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ satisfy (6.5). For $i = 1, 2$ let $u^i \in H^1(\Omega; \mathbb{R})$ be defined as before. Then*

$$\det[\nabla u^1 \ \nabla u^2] > 0 \quad \text{almost everywhere in } \Omega.$$

More precisely, $\log \det[\nabla u^1 \ \nabla u^2] \in \mathrm{BMO}(\Omega')$.

Here, BMO is the space of functions with bounded mean oscillations. Namely, a function $f \in L^1_{\mathrm{loc}}(\Omega'; \mathbb{R})$ is in $\mathrm{BMO}(\Omega')$ if

$$\sup_{Q \subseteq \Omega'} \frac{1}{|Q|} \int_Q \Big| f - \frac{1}{|Q|} \int_Q f \, \mathrm{d}y \Big| \mathrm{d}x < \infty,$$

where the sup is taken over all the squares Q contained in Ω'.

This theorem extends a well-known result for quasi-conformal mappings [**185**]. The proof is based on the ideas used in the previous section adapted to the non-smooth case via the concept of geometrical critical points [**14**]. The details are omitted, since a detailed presentation of the proof would require a separate chapter and would go beyond the scope of this book.

6.5. Absence of quantitative Jacobian bounds in three and higher dimensions

The Radó-Kneser-Choquet theorem and its generalisations to non-constant conductivities completely fail in three dimensions. Lewy's theorem is no longer true: there exists a harmonic homeomorphism of the unit three-dimensional sphere $\mathbb{B}^3$ into itself such that its Jacobian vanishes at some point. Furthermore, there exists a self-homeomorphism of $\mathbb{S}^2$ such that the Jacobian of its harmonic extension vanishes at some point, see *e.g.* [**215**], [**134**], [**64**], [**20**] and [**11**] for the study of critical points.
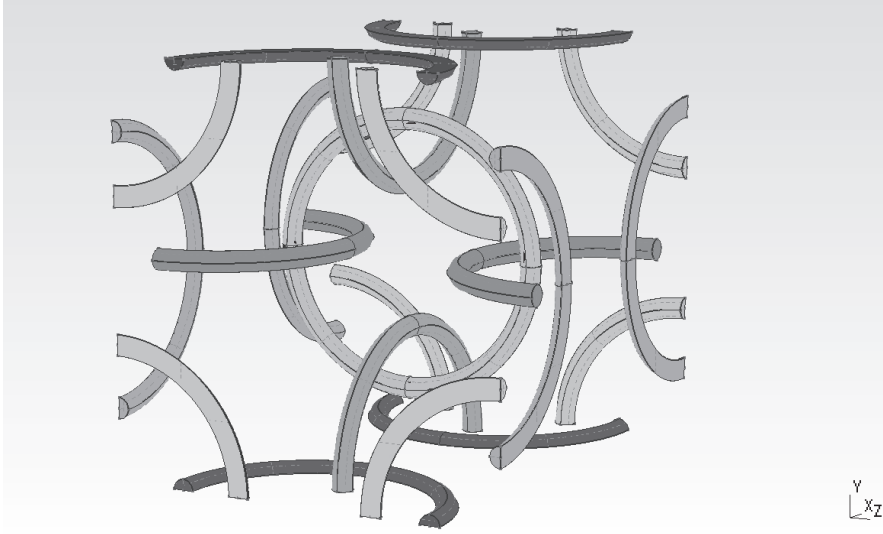
FIGURE 6.2. Sketch of the periodic set. The various colours were added for readability, to highlight which elements correspond to the same tori, once the cell is repeated periodically. The set Q is the union of all the rings.

In fact, using a result coming from the theory of homogenisation, it is possible to show that there is no good choice of boundary condition even locally. This result originally appeared in [**77**] in dimension three: we detail here a d-dimensional version $(d \geq 3)$, which is almost identical, and present some corollaries of this result.

Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain, for $d \geq 3$. Let $Y = [0,1]^3$ denote the unit cube, and a_3 be a piecewise constant Y-periodic function defined by

$$(6.9) \qquad a_3(y) = (\delta - 1)\mathbb{1}_Q(y) + 1, \quad y \in Y,$$

where $\mathbb{1}_Q$ is the characteristic function of Q. The set Q is made of rotations and translations of a scaled copy of the tori (that is, circular annuli whose cross-section is a disk), with cubic symmetry. An illustration of one such Q is given in Figure 6.2.

This type of construction was originally introduced in [**64**]; the variant presented here was introduced in [**63**] (see also [**120**]). The value of δ will be determined later. If $d \geq 4$, let a_d be a piecewise constant 1-periodic function such that in $[0,1]$ we have

$$a_d(t) = (\varkappa - 1)\, \mathbb{1}_{[0,\frac{1}{2})}(t) + 1, \quad t \in [0,1].$$

The constant $\varkappa > 0$, which depends on δ only, will also be determined later. Consider now the $(0,1)^d$-periodic conductivity $a \in L^\infty(\mathbb{R}^d; \mathbb{R})$ given by

$$(6.10) \qquad a(y_1, \ldots, y_d) = a_3(y_1, y_2, y_3) \prod_{i=4}^{d} a_d(y_i), \quad y \in [0,1]^d.$$

For $n \in \mathbb{N}$ and $i = 1, \ldots, d$ let $u_n^i \in H^1(\Omega; \mathbb{R})$ be the solution of

$$(6.11) \qquad \begin{cases} -\operatorname{div}(a(nx)\nabla u_n^i) = 0 & \text{in } \Omega, \\ u_n^i = \varphi^i & \text{on } \partial\Omega, \end{cases}$$

for some $\varphi^1, \ldots, \varphi^d \in H^{1/2}(\partial\Omega; \mathbb{R})$. Set $U_n = (u_n^1, \ldots, u_n^d)$ and $\varphi = (\varphi^1, \ldots, \varphi^d)$.

Note that U_n is bounded in $H^1(\Omega; \mathbb{R})^d$ independently of n, namely

$$\|U_n\|_{H^1(\Omega)^d} \leq C(\Omega, \varkappa, \delta)\|\varphi\|_{H^{1/2}(\partial\Omega)^d}.$$

Therefore, up to a subsequence, $U_n \rightharpoonup U^*$ weakly in $H^1(\Omega; \mathbb{R})^d$. It turns out that the whole sequence converges, and U^* satisfies a constant coefficient PDE. This is a so-called homogenisation result [**56**], [**167**], [**22**]. To state this result, we need to introduce an auxiliary periodic problem.

Definition 6.12. — The *periodic corrector matrix* $P \in L^2\big((0,1)^d; \mathbb{R}^{d \times d}\big)$ is given by $P_{ij} = \frac{\partial}{\partial y_i}\zeta_j$, where ζ is the solution of

$$\begin{cases} -\operatorname{div}(a\nabla\zeta) = 0 & \text{in } \mathbb{R}^d, \\ y \mapsto \zeta(y) - y & \text{is in } H^1_\#\big((0,1)^d\big), \\ \int_{(0,1)^d} \zeta(y)\,\mathrm{d}y = 0. \end{cases}$$

Here, the space $H^1_\#\big((0,1)^d\big)$ contains all $(0,1)^d$-periodic functions in $H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$.

Note that because of the structure of a, we have

$$(6.12) \qquad \zeta_j(y_1, \ldots, y_d) = \xi_j(y_1, y_2, y_3), \quad j = 1, 2, 3,$$

where $\xi \in H^1(Y; \mathbb{R}^3)$ is the solution of the three dimensional corrector problem, namely

$$\begin{cases} -\operatorname{div}(a_3 \nabla \xi) = 0 & \text{in } \mathbb{R}^3, \\ y \mapsto \xi(y) - y & \text{is in } H^1_\#(Y), \\ \int_Y \xi(y)\,\mathrm{d}y = 0, \end{cases}$$

and $\zeta_j(y_1, \ldots, y_d) = f(y_j)$ for $j > 3$, where $f \in H^1((0,1); \mathbb{R})$ is given by

$$\left(a_d f'\right)' = 0, \quad f(1) = f(0) + 1, \quad \int_0^1 f(t)\,\mathrm{d}t = 0.$$

Explicitly this gives

$$(6.13) \qquad P_{ij}(y) = f'(y_j)\,\delta_{ij} = \frac{2\varkappa}{1+\varkappa} \cdot \frac{1}{a_d(y_j)}\delta_{ij}, \quad j > 3 \text{ and } 1 \le i \le d,$$

and overall P_{ij} is block-diagonal with a 3×3 block for $1 \le i, j \le 3$, and diagonal for $i, j > 3$.

Theorem 6.13 (see [56], [167], [22]). — *The sequence $U_n = (u_n^1, \ldots, u_n^d)$ defined by (6.11) converges to U^* weakly in $H^1(\Omega; \mathbb{R}^d)$, where U^* satisfies*

$$\begin{cases} -\operatorname{div}(A^* \nabla U^*) = 0 & \text{in } \Omega, \\ U^* = \varphi & \text{on } \partial\Omega. \end{cases}$$

The constant matrix A^* is given by

$$(6.14) \qquad\qquad A^* = \int_{(0,1)^d} a(y)P(y)\,\mathrm{d}y,$$

where P is the corrector matrix given in Definition 6.12. Furthermore

$$(6.15) \qquad\qquad \nabla U_n - P(nx)\nabla U^* \longrightarrow 0 \text{ strongly in } L^1_{\mathrm{loc}}(\Omega).$$

In the case at hand, the matrix A^* has a specific form. Let us write for convenience $(y_1, y_2, y_3) = y'$. Inserting the explicit formula (6.13) in (6.14) we

find that for $d \geq j > 3$,

$$
\begin{aligned}
A_{ij}^* &= \delta_{ij} \int_{(0,1)^d} a_3(y') \prod_{\ell=4}^{d} a_d(y_\ell) \frac{2\varkappa}{1+\varkappa} \cdot \frac{1}{a_d(y_j)} \, \mathrm{d}y_1 \cdots \mathrm{d}y_d \\
&= \delta_{ij} \frac{2\varkappa}{1+\varkappa} \left(\int_Y a_3(y') \, \mathrm{d}y' \right) \left(\int_0^1 a_d(t) \, \mathrm{d}t \right)^{d-4} \\
&= \delta_{ij} \frac{2\varkappa}{1+\varkappa} \left((\delta-1)|Q| + 1 \right) \left(\tfrac{1}{2}(\varkappa+1) \right)^{d-4}.
\end{aligned}
$$

Note that (6.12) shows that when $j \leq 3$ and $4 \leq i$, $A_{ij}^* = 0$. Furthermore, for $1 \leq i, j \leq 3$

$$
\begin{aligned}
A_{ij}^* &= \int_Y a_3(y') \frac{\partial}{\partial y_i} \xi_j(y) \prod_{\ell=4}^{d} a_d(y_\ell) \, \mathrm{d}y \\
&= \left(\int_Y a_3(y') \frac{\partial}{\partial y_i} \xi_j(y') \, \mathrm{d}y' \right) \left(\tfrac{1}{2}(\varkappa+1) \right)^{d-3}.
\end{aligned}
$$

Because of the cubic symmetry satisfied by a_3, and the periodicity of $\xi_1 - y_1$,

$$
\int_Y a_3(y') \frac{\partial}{\partial y_i} \xi_1(y') \, \mathrm{d}y' = \delta_{1i} a_3^*.
$$

Furthermore, the cubic symmetry of a_3 also implies that

$$
\int_Y a_3(y') \frac{\partial}{\partial y_1} \xi_1(y') \, \mathrm{d}y' = \int_Y a_3(y') \frac{\partial}{\partial y_2} \xi_2(y') \, \mathrm{d}y' = \int_Y a_3(y') \frac{\partial}{\partial y_3} \xi_3(y') \, \mathrm{d}y'.
$$

As a consequence, A^* is diagonal and

$$
\left(\tfrac{1}{2}(\varkappa+1) \right)^{4-d} A_{ii}^* =
\begin{cases}
\frac{\varkappa+1}{2} a_3^* & \text{when } 1 \leq i \leq 3, \\
\frac{2\varkappa}{1+\varkappa} \left((\delta-1)|Q| + 1 \right) & \text{when } 3 < i \leq d.
\end{cases}
$$

Since P is the solution of a minimisation problem, one can establish that

$$
a_3^* = \int_Y a_3(y) P_{11}(y) \, \mathrm{d}y < \int_Y a_3(y) \, \mathrm{d}y = (\delta-1)|Q| + 1.
$$

Therefore, for any $\delta > 0$, there exists a unique $\varkappa \leq 1$ such that

(6.16)
$$
\frac{\varkappa+1}{2} a_3^* = \frac{2\varkappa}{1+\varkappa} \left((\delta-1)|Q| + 1 \right).
$$

With this choice of $\varkappa$ (as a function of δ), we find that A^* is a multiple of the identity matrix.

We then choose δ so that the periodic corrector matrix P has a positive determinant on a part of Y, and a negative determinant on another part. Such a choice is possible as it was shown in [64] in the case $d = 3$ – it readily applies in higher dimensions because of the structure of the matrix P.

Lemma 6.14 (*see* [64, Theorem 3]). — *There exist $\delta_0 > 0$, $\tau > 0$, Y_+ and Y_- open subsets of $Y \setminus \overline{Q}$ both of positive measure 2τ such that*

$$\det P \geq 2\tau \quad in \ Y_+ \quad and \quad \det P \leq -2\tau \ in \ Y_-.$$

We therefore fix δ according to Lemma 6.14 (and $\varkappa$ according to (6.16)). In view of Theorem 6.13 we have that U_n, the solution of

$$(6.17) \qquad \begin{cases} -\operatorname{div}\big(a(nx)\nabla U_n\big) = 0 & \text{in } \Omega, \\ U_n = \varphi & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$(6.18) \qquad \nabla U_n \longrightarrow P(nx)\nabla U^* \quad in \ L^1_{\mathrm{loc}}(\Omega),$$

where U^* satisfies

$$(6.19) \qquad \begin{cases} -\Delta U^* = 0 & \text{in } \Omega, \\ U^* = \varphi & \text{on } \partial\Omega. \end{cases}$$

Note that the asymptotic behaviour of ∇U_n given by (6.18) depends on two independent factors: P, whose determinant changes sign locally and was constructed independently of Ω and φ, and ∇U^*, the harmonic lift of the boundary condition φ in Ω. It is clear that the variations of the sign of $\det(\nabla U_n)$ cannot be fully controlled by the boundary condition, which only acts on U^*. The main result of this section is a quantitative version of this statement.

Theorem 6.15 (*see* [77]). — *Let $\Omega' \Subset \Omega$ be a smooth domain. Given $\rho > 0$, $x_0 \in \Omega'$ such that $B_\rho(x_0) \subseteq \Omega'$ and $\lambda > 0$ let*

$$A_d(x_0, \rho, \lambda) := \big\{ \varphi \in H^{1/2}(\partial\Omega; \mathbb{R}^d) : \det(\nabla U^*) \geq \lambda \|\varphi\|^d_{H^{1/2}(\partial\Omega)} \ in \ B_\rho(x_0) \big\},$$

where U^* is the harmonic extension of φ given by (6.19). There exist n, depending only on ρ, Ω, Ω' and λ, a universal constant $\tau > 0$ and two open subsets B_+ and B_- of $B_\rho(x_0)$ such that

$$|B_+| \geq \tau \big| B_\rho(x_0) \big| \quad and \quad |B_-| \geq \tau \big| B_\rho(x_0) \big|,$$

and for all $\varphi \in A_d(x_0, \rho, \lambda)$, there holds

$$\det(\nabla U_n)(x) \leq -\tau\lambda\|\varphi\|^d_{H^{1/2}(\partial\Omega)} \qquad in \ B_-,$$

$$\det(\nabla U_n)(x) \geq \tau\lambda\|\varphi\|^d_{H^{1/2}(\partial\Omega)} \qquad in \ B_+,$$

where U_n is the solution of (6.17).

Remark. — The pathological conductivity used here was constructed to be piecewise constant, but a standard mollification argument allows us to consider a smooth approximation of a instead to construct similar counterexamples.

To prove this result we need a quantitative convergence estimate in lieu of (6.18).

This follows from a regularity result. Since the conductivity γ is piecewise constant (and therefore piecewise smooth), and the set Q has C^∞ smooth boundaries (and therefore $C^{1,\alpha}$ smooth boundaries), the regularity results [**142**], [**141**] show that U_n is also piecewise $C^{1,\beta}$ for some $\beta > 0$, up to the boundary of the set Q in Ω'. In fact, this provides uniform $W^{1,\infty}$ estimates for U_n, independently of n (see [**141**]). This result has been then successfully expanded to provide error estimate results for U_n, see [**55**], [**148**].

Lemma 6.16 (*see* [**141**, Theorem 3.4], [**55**, Theorem 3.6] *or* [**148**, Theorem 4.2]). — *There exists a constant $C > 0$ depending only on Ω, Ω', Q, δ and $\varkappa$ such that*

$$\|\nabla U_n\|_{L^\infty(\Omega')} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)},$$

$$\|P(nx)\|_{L^\infty(\Omega')} \leq C,$$

$$\|\nabla U_n - P(nx)\nabla U^*\|_{L^\infty(\Omega')} \leq C\frac{\|\varphi\|_{H^{1/2}(\partial\Omega)}}{n^{1/3}}.$$

Proof of Theorem 6.15. — In Ω', we have

$$\det(\nabla U_n) = \det\big(P(nx)\nabla U^*\big) + R_n = \det\big(P(nx)\big)\det(\nabla U^*) + R_n,$$

with

$$\|R_n\|_{L^\infty(\Omega')} \leq C\big\|\nabla U_n - P(nx)\nabla U^*\big\|_{L^\infty(\Omega')}$$
$$\times \big(\|\nabla U_n\|_{L^\infty(\Omega')} + \|P(nx)\nabla U^*\|_{L^\infty(\Omega')}\big)^{d-1},$$

and thanks to Lemma 6.16,

$$\|R_n\|_{L^\infty(\Omega')} \le C \frac{\|\varphi\|^d_{H^{1/2}(\partial\Omega)}}{n^{1/3}}.$$

Let $B_\pm = \{x \in B(x_0, \rho) : nx \in Y_\pm + \mathbb{Z}^d\}$. For n large enough, $|B_\pm| \ge \tau|B(x_0, \rho)|$. Thanks to Lemma 6.14, we have $\pm \det(P(nx)) \ge 2\tau$ in $B_\pm$, therefore

$$\pm \det(\nabla U_n) \ge 2\tau \det(\nabla U^*) - \frac{C}{n^{1/3}}\|\varphi\|^d_{H^{1/2}(\partial\Omega)}.$$

With $\varphi \in A_d(x_0, \rho, \lambda)$, this implies

$$\pm \det(\nabla U_n) \ge \left(2\tau\lambda - \frac{C}{n^{1/3}}\right)\|\varphi\|^d_{H^{1/2}(\partial\Omega)} \ge \tau\lambda\|\varphi\|^d_{H^{1/2}(\partial\Omega)},$$

for $n^{1/3}\tau\lambda \ge C$. □

The next corollary highlights that even locally no set of boundary conditions can be chosen *a priori*, that is, independently of the unknown conductivity, to enforce a positivity constraint on the Jacobian. The periodic conductivity a is equal to a_3 given by (6.9), with δ chosen appropriately as above, with the same cubic symmetry.

Definition 6.17. — Given $\Omega \subseteq \mathbb{R}^3$ a smooth bounded domain, and $\varphi_1, \varphi_2, \varphi_3$ in $H^{1/2}(\partial\Omega; \mathbb{R})$, we say that the harmonic extension of $(\varphi_1, \varphi_2, \varphi_3)$ *has maximal rank* in Ω if the solution of

$$\begin{cases} \Delta U^* = 0 & \text{in } \Omega, \\ U^* = (\varphi_1, \varphi_2, \varphi_3) & \text{on } \partial\Omega, \end{cases}$$

is such that $\det \nabla U^*(z) \ne 0$ for some $z \in \Omega$.

Clearly, a choice of boundary values φ_1, φ_2 and φ_3 whose harmonic extension does not have maximal rank in Ω will never be suitable for our purposes, since the Jacobian constraint is not satisfied even for the trivial conductivity $a \equiv 1$.

Corollary 6.18. — *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz bounded domain, and $\Omega' \Subset \Omega$ be a smooth subdomain. Take $\varphi_1, \ldots, \varphi_N$ in $H^{1/2}(\partial\Omega; \mathbb{R})$ for some $N \in \mathbb{N}^*$. For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that the following is true.*

For every open ball $B_\varepsilon \subseteq \Omega'$ of radius ε, there exists $x_1 \in B_\varepsilon$ such that

$$\max_{1 \leq i,j,k \leq N} \left| \det \left([\nabla u_i(x_1), \nabla u_j(x_1), \nabla u_k(x_1)] \right) \right| \leq \varepsilon,$$

where u_i is the solution of

$$\begin{cases} \operatorname{div}(a(nx)\nabla u_i) = 0 & in \ \Omega, \\ u_i = \varphi_i & on \ \partial\Omega. \end{cases}$$

Furthermore, for every open ball $B_\varepsilon \subseteq \Omega'$ of radius ε and every $1 \leq i,j,k \leq N$ such that the harmonic extension of $(\varphi_i, \varphi_j, \varphi_k)$ has maximal rank in Ω there exists $x_2 \in B_\varepsilon$ such that

$$\det \left([\nabla u_i(x_2), \nabla u_j(x_2), \nabla u_k(x_2)] \right) = 0.$$

Proof. — Let Ω'' be a smooth domain such that $\Omega' \Subset \Omega'' \Subset \Omega$. Following the proof of Theorem 6.15, for any $1 \leq i,j,k \leq N$, we have

$$(6.20) \qquad \det \left(\nabla U_n(x) \right) = \det \left(P(nx) \right) \det \left(\nabla U^*(x) \right) + R_n(x), \quad x \in \Omega'',$$

where $U_n = (u_i, u_j, u_k)$ and

$$\begin{cases} \Delta U^* = 0 & in \ \Omega, \\ U^* = (\varphi_i, \varphi_j, \varphi_k) & on \ \partial\Omega, \end{cases}$$

with

$$(6.21) \qquad \left| R_n(x) \right| \leq \frac{C}{n^{1/3}} \quad and \quad \left| \det(\nabla U^*(x)) \right| \leq C \quad for \ all \ x \in \Omega'',$$

for some $C > 0$ depending only on $\max_i \|\varphi_i\|_{H^{1/2}(\partial\Omega)}$. Therefore, for any

$$n \geq n_0 = \left(\frac{2}{\varepsilon} C \right)^3,$$

we have

$$(6.22) \qquad \left| R_n(x) \right| \leq \tfrac{1}{2}\varepsilon, \quad x \in \Omega''.$$

Let $\{B(x_p, \tfrac{1}{3}\varepsilon)\}_{1 \leq p \leq P}$ be a finite cover of Ω' with balls of radius $\tfrac{1}{3}\varepsilon$ such that $B(x_p, \tfrac{1}{3}\varepsilon) \cap \Omega'' \neq \varnothing$. Let S denote the set of all triples $(i,j,k) \in \{1,\dots,N\}^3$ such that the harmonic extension of $(\varphi_i, \varphi_j, \varphi_k)$ has maximal rank in Ω, namely

$$S = \left\{ (i,j,k) \in \{1,\dots,N\}^3 : \det \nabla U^*(z) \neq 0 \text{ for some } z \in \Omega \right\}.$$

Since $\det \nabla U^*$ is analytic in Ω for every (i,j,k), for every $p = 1, \ldots, P$ there exists $z_p \in B(x_p, \frac{1}{3}\varepsilon) \cap \Omega''$ such that $\det \nabla U^*(z_p) \neq 0$ for every $(i,j,k) \in S$. By continuity, for every p there exists a ball $B(z_p, \eta_p) \subseteq B(x_p, \frac{1}{3}\varepsilon) \cap \Omega''$ with $\eta_p > 0$ and a constant $c_p > 0$ such that for any $(i,j,k) \in S$

$$\det \nabla U^*(x) \det \nabla U^*(z_p) \geq c_p, \quad x \in B(z_p, \eta_p).$$

Note that we may choose a common radius $\eta > 0$ and lower bound $c > 0$ for this finite collection of balls, namely

(6.23) $$\det \nabla U^*(x) \det \nabla U^*(z_p) \geq c, \quad (i,j,k) \in S, x \in B(z_p, \eta).$$

By Lemma 6.14, there exists a universal constant $\tau > 0$, and two open balls, $B_+, B_- \subseteq Y \setminus \overline{Q}$, such that

(6.24) $$\inf_{y \in B_+} \det P(y) \geq 2\tau, \quad \sup_{y \in B_-} \det P(y) \leq -2\tau.$$

As a consequence, there exists an open ball $B_0 \subseteq Y$ such that

(6.25) $$\sup_{y \in B_0} |\det P(y)| \leq \frac{\varepsilon}{2C}.$$

Let $n_1 = (C^2/c\tau)^3$, so that by (6.21) for every $n \geq n_1$ we have

$$\left| R_n(x) \det \nabla U^*(z) \right| \leq c\tau, \quad x, z \in \Omega''.$$

Thus, in view of (6.23) and (6.24), for all $n \geq n_1$, $(i,j,k) \in S$, $p = 1, \ldots, P$ and $x \in B(z_p, \eta)$ we have

(6.26) $$\begin{cases} \inf_{y \in B_+} \left(\det P(y) \right) \det \nabla U^*(x) \det \nabla U^*(z_p) - |R_n(x) \det \nabla U^*(z_p)| \geq \tau c, \\ \sup_{y \in B_-} \left(\det P(y) \right) \det \nabla U^*(x) \det \nabla U^*(z_p) + |R_n(x) \det \nabla U^*(z_p)| \leq -\tau c. \end{cases}$$

For $n \in \mathbb{N}$ and $p = 1, \ldots, P$, let

$$B_+^n(p) = \left\{ x \in B(z_p, \eta) : nx \in B_+ + \mathbb{Z}^3 \right\},$$
$$B_-^n(p) = \left\{ x \in B(z_p, \eta) : nx \in B_- + \mathbb{Z}^3 \right\},$$
$$B_0^n(p) = \left\{ x \in B(z_p, \eta) : nx \in B_0 + \mathbb{Z}^3 \right\}.$$

Choose $n \geq \max(n_0, n_1)$ large enough so that $B_+^n(p) \neq \varnothing$, $B_-^n(p) \neq \varnothing$ and $B_0^n(p) \neq \varnothing$ for all $1 \leq p \leq P$. We will now show that n is an appropriate choice to satisfy the claims of the corollary.

Given a ball $B_\varepsilon \subseteq \Omega'$ of radius ε, there exists at least one $p \in \{1, \ldots, P\}$ such that $B(x_p, \frac{1}{3}\varepsilon) \subseteq B_\varepsilon$. Pick any $x_1 \in B_0^n(p) \subseteq B_\varepsilon$. If $(i, j, k) \in S$, thanks to (6.20), (6.22) and (6.25) there holds

$$\left| \det(\nabla U_n(x_1)) \right| \leq \left| \det(P(nx_1)) \right| \cdot \left| \det(\nabla U^*(x_1)) \right| + \left| R_n(x_1) \right| \leq \varepsilon.$$

If $(i, j, k) \notin S$ then by (6.20) and (6.22) we have

$$\left| \det(\nabla U_n(x_1)) \right| = \left| R_n(x_1) \right| \leq \tfrac{1}{2}\varepsilon \leq \varepsilon,$$

and so the first part of the statement follows.

Turning to the second statement of the corollary, given $(i, j, k) \in S$, choose $x_+ \in B_+^n(p)$ and $x_- \in B_-^n(p)$. By construction, there exists a continuous path $\gamma \colon [0, 1] \to B(z_p, \eta)$ such that $\gamma(0) = x_+$, $\gamma(1) = x_-$ and $n\gamma(t) \in (Y \setminus \overline{Q}) + \mathbb{Z}^3$ for every $t \in [0, 1]$. Thus, the function

$$g \colon t \longmapsto \det \nabla U_n\big(\gamma(t)\big) \det \nabla U^*(z_p),$$

is continuous. Further, in view of (6.26) it satisfies $g(0) \geq \tau c$ and $g(1) \leq -\tau c$. By the intermediate value theorem, there exists $x_2 \in B(z_p, \eta) \subseteq B_\varepsilon$ such that $\det(\nabla U_n(x_2)) \det \nabla U^*(z_p) = 0$. Thus, by (6.23), we have $\det(\nabla U_n(x_2)) = 0$, as desired. $\qquad\qquad\square$

CHAPTER 7

COMPLEX GEOMETRIC OPTICS SOLUTIONS AND THE RUNGE APPROXIMATION PROPERTY

7.1. Introduction

In Chapter 6 we studied the boundary control of d solutions to the conductivity equation

$$(7.1) \qquad -\operatorname{div}(a\nabla u^i) = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^d,$$

in order to enforce a non-vanishing Jacobian constraint inside the domain. We saw that some generalisations of the Radó-Kneser-Choquet theorem completely solved the problem in two dimensions. Moreover, we exhibited a counter-example indicating that such a result cannot hold in more than two dimensions without *a priori* information concerning the conductivity a.

In any dimension, similar results cannot be obtained with solutions of equations of the form

$$(7.2) \qquad \operatorname{div}(a\nabla u^i) + q u^i = 0 \quad \text{in } \Omega,$$

for some unknown $q \in L^\infty(\Omega)$, $q \geq 0$ almost everywhere. Indeed, one fundamental ingredient of the proof of those results was the maximum principle, which does not hold in general for PDE of this type.

This chapter is devoted to the discussion of two possible strategies that have been used to overcome these issues: complex geometric optics (CGO) solutions and the Runge approximation property. These two techniques are very

different but both are based on a common idea, namely approximating so-
lutions of (7.1) (or (7.2)) by solutions to constant coefficient equations, for
which explicit solutions can be constructed. These approaches can be applied
provided that the oscillations of the parameters of the equations are known to
be bounded *a priori*.

Consider for simplicity the conductivity equation (7.1) with scalar coefficient
a in three dimensions. In the constant case, that is, with $a = 1$, we are left with
the Laplace equation

$$-\Delta u^i = 0 \quad \text{in} \ \ \Omega.$$

The starting point of the CGO approach consists of taking solutions to this
equation of the form $u_\rho(x) = e^{\rho \cdot x}$ for some $\rho \in \mathbb{C}^3$ such that $\rho \cdot \rho = 0$ (where
$\rho_1 \cdot \rho_2 := (\operatorname{Re} \rho_1 \cdot \operatorname{Re} \rho_2 - \operatorname{Im} \rho_1 \cdot \operatorname{Im} \rho_2) + i(\operatorname{Re} \rho_1 \cdot \operatorname{Im} \rho_2 + \operatorname{Im} \rho_1 \cdot \operatorname{Re} \rho_2))$,
namely harmonic complex plane waves, as it was done in the seminal paper
of Calderón [**70**]. With suitable choices for the parameter ρ, it is possible to
satisfy the desired condition $|\det[\nabla u^1 \nabla u^2 \nabla u^3]| > 0$. It remains to show that
the solutions in the general case with non-constant coefficients can be approxi-
mated by the solutions u_ρ as $|\rho| \to \infty$. This result was originally proved in [**202**]
and mainly applied to inverse boundary value problems [**207**]. Regularity esti-
mates adapted to hybrid problems were derived in [**49**]. CGO solutions have
been widely used in hybrid imaging techniques to exhibit solutions satisfying
several local non-zero constraints [**205**], [**49**], [**41**], [**46**], [**27**], [**124**], [**161**],
[**42**], [**160**], [**40**], [**51**], [**28**], [**48**], [**43**]. A detailed discussion of this strategy
is presented in Section 7.2, omitting the existence and regularity results for
the CGO solutions.

Let us now turn to the Runge approximation approach. It is enough to
choose the functions $u^i = x_i$ as solutions in the constant coefficient case. With
this choice there holds

$$\det[\nabla u^1 \ \nabla u^2 \ \nabla u^3] = \det[\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3] = 1,$$

and so the desired constraint is satisfied everywhere. By the Runge approxi-
mation property, it is possible to construct solutions to the non-constant co-
efficient PDE which approximate x_i in a given small ball inside the domain.

By covering the domain Ω with a finite number of these small balls the desired constraint is enforced globally. The Runge approximation property for PDE dates back to the 1950s, [150], [136]. It was applied to hybrid imaging problems in [51], which we follow in the presentation given in Section 7.3 (see also [43], [162], [48]). The main advantages of this approach over the CGO approach are:

1) It is applicable with any second order elliptic equation, and in particular with anisotropic leading order coefficients, while CGO solutions can be constructed only with isotropic coefficients;
2) While the CGO estimates require high regularity assumptions on the coefficients, the Runge approximation property holds provided that the PDE enjoys the unique continuation property;
3) Since any solutions to the constant coefficient case can be approximated, more general non-zero constraints can be satisfied.

However, there is a price to pay. While the CGO solutions give a non-vanishing Jacobian globally inside the domain for a single (complex) choice of the boundary conditions, by using the Runge approximation property the constraint under consideration holds only locally in fixed small balls. Therefore, many different boundary conditions must be used to cover the whole domain. Moreover, the CGO solutions are explicitly constructed (depending on the coefficients), while the Runge approximation approach only gives a theoretical existence result of suitable boundary conditions.

7.2. Complex geometric optics solutions

7.2.1. Harmonic complex plane waves. — The starting point of the CGO approach is always the Laplace equation

$$-\Delta u = 0 \quad \text{in} \ \ \Omega.$$

The CGO solutions are approximations in the non-constant coefficient case to the harmonic complex plane waves of the form $u_\rho(x) = e^{\rho \cdot x}$ for some $\rho \in \mathbb{C}^3$ such that $\rho \cdot \rho = 0$. These are harmonic functions in the whole space, since

$$\Delta u_\rho(x) = \operatorname{div}(\rho e^{\rho \cdot x}) = \rho \cdot \rho \, e^{\rho \cdot x} = 0, \quad x \in \mathbb{R}^d.$$

Let us now explain why these solutions are of interest for us. As it was shown in [**40**], suitable choices of the parameter ρ allow to satisfy the constraint $|\det[\nabla u^1 \, \nabla u^2 \, \nabla u^3]| > 0$. Indeed, take $\rho_1 = t(\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2)$ and $\rho_2 = t(\mathbf{e}_3 + \mathrm{i}\mathbf{e}_1)$ for some $t > 0$ and consider the solutions $u^1 = \operatorname{Re} u_{\rho_1}$, $u^2 = \operatorname{Im} u_{\rho_1}$ and $u^3 = u_{\rho_2}$. A direct calculation gives

$$(7.3) \qquad J(x) := \det \left[\nabla \operatorname{Re} u_{\rho_1} \ \nabla \operatorname{Im} u_{\rho_1} \ \nabla u_{\rho_2} \right](x)$$
$$= t^2 \, \mathrm{e}^{t(2x_1 + x_3 + \mathrm{i}x_1)} \det[\mathbf{e}_1 \ \mathbf{e}_2 \ \rho_2] = t^3 \, \mathrm{e}^{t(2x_1 + x_3 + \mathrm{i}x_1)}.$$

We have obtained the condition $|J(x)| > 0$. We have used two real solutions and one complex solution. However, only real solutions (and real illuminations) can be considered in practice. In order to overcome this problem, it is enough to choose the real solutions $u^{3,1} = \operatorname{Re} u_{\rho_2}$ and $u^{3,2} = \operatorname{Im} u_{\rho_2}$. Since u^1 and u^2 are real, there holds

$$\operatorname{Re} J = \det \left[\nabla u^1 \ \nabla u^2 \ \nabla u^{3,1} \right], \quad \operatorname{Im} J = \det \left[\nabla u^1 \ \nabla u^2 \ \nabla u^{3,2} \right].$$

Therefore, since $|J(x)| > 0$ everywhere, we obtain the decomposition

$$\Omega = \Omega^1 \cup \Omega^2,$$

where we have set $\Omega^j = \{x \in \Omega : |\det[\nabla u^1 \ \nabla u^2 \ \nabla u^{3,j}](x)| > 0\}$.

Thus, in the harmonic case we can construct suitable illuminations whose corresponding solutions to the Laplace equation deliver a non-vanishing Jacobian everywhere, in the sense made precise above. It remains to show that these solutions can be approximated in the general case with non-constant coefficient.

7.2.2. The main result. — Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and, as in [**42**], we consider the elliptic equation

$$\begin{cases} -\operatorname{div}(a\nabla u) + qu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $a \in W^{2,\infty}(\Omega; \mathbb{R})$, $q \in L^\infty(\Omega; \mathbb{C})$ and a satisfies the uniform ellipticity condition

$$(7.4) \qquad \Lambda^{-1} \leq a \leq \Lambda \quad \text{almost everywhere in } \Omega.$$

After the so-called Liouville change of unknown $v = \sqrt{a}\,u$, we see that v satisfies

$$-\Delta v + \Big(\frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{q}{a}\Big)v = 0 \quad \text{in } \Omega.$$

As a consequence, setting $q' = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{q}{a}$ and considering the coefficient q' as defined on the whole space $\mathbb{R}^d$ and with compact support, it is sufficient to study the problem for the simplified Schrödinger-type equation

$$(7.5) \qquad\qquad -\Delta v + q'v = 0 \quad \text{in } \mathbb{R}^d.$$

In this form, it is clear that the case with $q' \neq 0$ can be considered as a lower order perturbation of the Laplace equation, for which we constructed simple solutions, the harmonic complex plane waves of the form $e^{\rho\cdot x}$. Thus, it is natural to seek solutions to (7.5) as perturbations of these plane waves, namely of the form

$$(7.6) \qquad\qquad v_\rho(x) = e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big), \quad x \in \mathbb{R}^d$$

for some $\rho \in \mathbb{C}^d$ such that $\rho \cdot \rho = 0$, where ψ_ρ is an error term, due to the presence of the perturbation q'. The functions v_ρ are called complex geometric optics solutions. Existence and regularity of these solutions are guaranteed provided that the coefficient q' is smooth enough.

Theorem 7.1 (see [202, 49]). — *Take $\varkappa \in \mathbb{N}^*$, $\delta > 0$, $\rho \in \mathbb{C}^d$ with $\rho \cdot \rho = 0$ and let $q' \in H^{d/2+\varkappa+\delta}(\mathbb{R}^d; \mathbb{C})$ be compactly supported. There exists $\eta > 0$ such that if $|\rho| \geq \eta$ there exists $\psi_\rho \in H^{d/2+\varkappa+1+\delta}(\mathbb{R}^d; \mathbb{C})$ such that v_ρ defined as in (7.6) is a solution to (7.5). Moreover, for some $C > 0$,*

$$(7.7) \qquad\qquad \|\psi_\rho\|_{C^\varkappa(\overline{\Omega})} \leq \frac{C}{|\rho|}.$$

The (complete) proof of this result is much beyond the scope of these notes. Not only does this result guarantee the existence of CGO solutions, but furthermore it gives the approximation property (7.7). This is the property we referred to at the beginning of this chapter: in view of (7.6), as $|\rho| \to \infty$, the CGO solutions approach the harmonic plane waves of the form $e^{\rho\cdot x}$. As observed in §7.2.1, harmonic plane waves can be used to satisfy local non-zero

constraints. Therefore, CGO solutions can be used to enforce the same non-zero constraints in the general case, with non-constant coefficients, as discussed in detail below.

7.2.3. Boundary control to enforce non-zero constraints

7.2.3.1. *The Jacobian of solutions to the conductivity equation.* — Inspired by Chapter 6, we first consider the conductivity equation in three dimensions,

$$(7.8) \qquad \begin{cases} -\operatorname{div}(a\nabla u^i) = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain of class $C^{1,\alpha}$ and a satisfies (7.4), and look for real boundary values φ_1, φ_2 and φ_3 such that $|\det[\nabla u^1 \ \nabla u^2 \ \nabla u^3]| > 0$ at least locally in Ω. The main result reads as follows.

Theorem 7.2 (see [40]). — *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $C^{1,\alpha}$ domain and $a \in H^{3/2+3+\delta}(\mathbb{R}^3; \mathbb{R})$ satisfy (7.4). There exists an open set of boundary conditions $(\varphi_1, \dots, \varphi_4) \in C^2(\overline{\Omega}; \mathbb{R})^4$ such that*

$$(7.9) \qquad \left| \det\left[\nabla u^1 \ \nabla u^2 \ \nabla u^3\right](x)\right| + \left| \det\left[\nabla u^1 \ \nabla u^2 \ \nabla u^4\right](x)\right| > 0, \quad x \in \overline{\Omega},$$

where u^i is the unique solution to (7.8).

Before proving this result, some comments are in order:

▷ By Sobolev embedding, the coefficient a belongs to $C^3(\overline{\Omega}; \mathbb{R})$. As announced in the introduction, regularity of the coefficient is required for this method to work.

▷ Compared to the results given in Chapter 6, when $d = 3$, four different boundary values have to be taken and the non-degeneracy condition only holds locally for three fixed solutions (see §6.5). However, we have the following decomposition

$$\Omega = \Omega^1 \cup \Omega^2,$$

where we have set $\Omega^j = \{x \in \Omega : |\det[\nabla u^1 \ \nabla u^2 \ \nabla u^{2+j}](x)| > 0\}$.

▷ Theorem 7.2 applies to (a mollified version of) the sequence of microstructures $a(n \bullet)$ introduced in Section 6.5, for any $n > 0$. However,

Theorem 6.15 indicates that the number of connected components in Ω^1 and Ω^2 will increase with n, and the positive lower bound will decrease with n.

Proof. — Without loss of generality, we can assume that $a = 1$ outside of a ball containing Ω. In the notation of Theorem 7.1, this implies that $q' = \Delta\sqrt{a}/\sqrt{a}$ belongs to $H^{5/2+\delta}(\mathbb{R}^3; \mathbb{R})$, so that all the assumptions of the theorem are satisfied with $\varkappa = 1$.

As in §7.2.1, set $\rho_1 = t(e_1 + \mathrm{i}e_2)$ and $\rho_2 = t(e_3 + \mathrm{i}e_1)$ for some $t > 0$. In view of Theorem 7.1, for t big enough and $\ell = 1, 2$ there exist solutions u_{ρ_ℓ} to (7.8) in $\mathbb{R}^3$ such that

$$u_{\rho_\ell}(x) = a^{-\frac{1}{2}}\,\mathrm{e}^{\rho_\ell \cdot x}\big(1 + \psi_{\rho_\ell}(x)\big), \quad x \in \mathbb{R}^3.$$

Differentiating this identity, and taking into account (7.7) and that $a \in C^1(\overline{\Omega})$, we obtain for $\ell = 1, 2$ as $t \to \infty$

$$\nabla u_{\rho_\ell}(x) = a^{-\frac{1}{2}}\,\mathrm{e}^{\rho_\ell \cdot x}\big(\rho_\ell + O(1)\big), \quad x \in \overline{\Omega},$$

where the constant hidden in the O symbol is independent of x and t. As in §7.2.1, it remains to calculate the Jacobian of the map $(\mathrm{Re}\,u_{\rho_1}, \mathrm{Im}\,u_{\rho_1}, u_{\rho_2})$. Arguing as in (7.3), a straightforward computation shows that for every $x \in \overline{\Omega}$

$$J(x) := \det\big[\nabla\mathrm{Re}\,u_{\rho_1}\ \nabla\mathrm{Im}\,u_{\rho_1}\ \nabla u_{\rho_2}\big](x) = t^3\mathrm{e}^{t(2x_1+x_3+\mathrm{i}x_1)}\big(1 + O(t)\big).$$

Choosing now t big enough so that $|O(t)| \leq \frac{1}{2}$ we obtain

$$\big|\det\big[\nabla\mathrm{Re}\,u_{\rho_1}\ \nabla\mathrm{Im}\,u_{\rho_1}\ \nabla u_{\rho_2}\big](x)\big| > 0, \quad x \in \overline{\Omega}.$$

Taking real and imaginary parts yields for every $x \in \overline{\Omega}$

$$\big|\det\big[\nabla\mathrm{Re}\,u_{\rho_1}\ \nabla\mathrm{Im}\,u_{\rho_1}\ \nabla\mathrm{Re}\,u_{\rho_2}\big](x)\big| + \big|\det[\nabla\mathrm{Re}\,u_{\rho_1}\ \nabla\mathrm{Im}\,u_{\rho_1}\ \nabla\mathrm{Im}\,u_{\rho_2}\big](x)\big| > 0.$$

Hence (7.9) is immediately verified setting

$$\varphi_1 = \mathrm{Re}\,u_{\rho_1|\partial\Omega}, \quad \varphi_2 = \mathrm{Im}\,u_{\rho_1|\partial\Omega}, \quad \varphi_3 = \mathrm{Re}\,u_{\rho_2|\partial\Omega}, \quad \varphi_4 = \mathrm{Im}\,u_{\rho_2|\partial\Omega}.$$

Finally, standard elliptic regularity theory [104] ensures the continuity of the map $\varphi \in C^2(\overline{\Omega}; \mathbb{R}) \mapsto u \in C^1(\overline{\Omega}; \mathbb{R})$, so that (7.9) still holds true for an open set of boundary values in $C^2(\overline{\Omega}; \mathbb{R})^4$ near $\varphi_1 = \mathrm{Re}\,u_{\rho_1|\partial\Omega}$, $\varphi_2 = \mathrm{Im}\,u_{\rho_1|\partial\Omega}$, $\varphi_3 = \mathrm{Re}\,u_{\rho_2|\partial\Omega}$ and $\varphi_4 = \mathrm{Im}\,u_{\rho_2|\partial\Omega}$. $\square$

7.2.3.2. *The Schrödinger equation.* — Let $\Omega \subseteq \mathbb{R}^d$ be a bounded $C^{1,\alpha}$ domain for $d = 2$ or $d = 3$. We consider here the Schrödinger equation

$$(7.10) \qquad \begin{cases} -\Delta u^i + q u^i = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $q \in L^\infty(\Omega; \mathbb{C})$ is such that

$$(7.11) \qquad 0 \text{ is not an eigenvalue of } (-\Delta + q) \text{ in } \Omega.$$

A more general second order elliptic equation with non-constant leading order term could be considered as well, as in §7.2.2. We omit this generalisation which would make the exposition slightly more involved. Such an extension will not be needed in the applications discussed in Part II.

We look for $d + 1$ (complex) boundary conditions φ_i such that the corresponding solutions to the above equation satisfy for every $x \in \overline{\Omega}$

$$(7.12a) \qquad \left| u^1(x) \right| > 0,$$

$$(7.12b) \qquad \left| \det \left[\nabla u^2 \; \cdots \; \nabla u^{d+1} \right](x) \right| > 0,$$

$$(7.12c) \qquad \left| \det \left[\begin{matrix} u^1 & \cdots & u^{d+1} \\ \nabla u^1 & \cdots & \nabla u^{d+1} \end{matrix} \right](x) \right| > 0.$$

The use of complex boundary values allows each of these conditions to be satisfied in the whole domain. Should only real boundary conditions be allowed, real and imaginary parts would have to be taken, as it was done previously in the case of the Jacobian constraint for the conductivity equation. In such a case, the above constraints would be satisfied only locally in the domain (as in (7.9)).

These constraints are motivated by the hybrid problems we shall discuss in Part II, and should be satisfied simultaneously. They somehow complete the Jacobian constraint given in (7.12b), which has been previously considered.

In particular, (7.12a) refers to the availability of one non-vanishing solution. This is certainly the simplest constraint one could think of, and naturally appears in several hybrid problems where the internal data depend on the solutions u^i and not on their gradients. The reason why this constraint did not appear in the study of the conductivity equation is evident: the maximum principle gives it for free, provided that the boundary value has a constant sign. On the other hand, this condition cannot be taken for granted for solutions

of (7.10). Indeed, this PDE models wave phenomena, and as such its solutions typically have an oscillatory behaviour. The third constraint (7.12c) is an "augmented" Jacobian: it requires the availability of $d + 1$ independent measurements.

We now prove that the above constraints are satisfied in the whole domain $\overline{\Omega}$ by suitable CGO solutions.

Theorem 7.3. — *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded $C^{1,\alpha}$ domain for $d = 2$ or $d = 3$ and let $q \in H^{d/2+1+\delta}(\mathbb{R}^d; \mathbb{C})$ satisfy (7.11). There exists an open set of boundary conditions $(\varphi_1, \ldots, \varphi_{d+1}) \in C^2(\overline{\Omega}; \mathbb{C})^{d+1}$ such that the constraints in (7.12) are verified for every $x \in \overline{\Omega}$, where u^i is the unique solution to (7.10).*

Proof. — Without loss of generality, we assume that $q = 0$ outside of a ball containing Ω, so that all the assumptions of Theorem 7.1 are satisfied for $q' = q$ and $\varkappa = 1$. Set now

$$\rho_1 = \tfrac{1}{2}t(e_1 + ie_2), \quad \rho_i = t(e_{i-1} + ie_i), \quad i = 2, d, \quad \rho_{d+1} = t(e_d + ie_1).$$

Note that $\rho_i \cdot \rho_i = 0$ and that $|\rho_i| \geq t/\sqrt{2}$ for every $i = 1, \ldots, d + 1$. Thus, in view of Theorem 7.1, for every i and for t big enough there exist solutions u_{ρ_i} to $-\Delta u + qu = 0$ in $\mathbb{R}^d$ such that

$$u_{\rho_i}(x) = e^{\rho_i \cdot x}\big(1 + \psi_{\rho_i}(x)\big), \quad x \in \mathbb{R}^3,$$

where the error functions $\psi_{\rho_i} \in C^2(\mathbb{R}^d)$ satisfy the bounds $\|\psi_{\rho_i}\|_{C^1(\overline{\Omega})} \leq C t^{-1}$ for some positive constant $C > 0$. There holds

$$(7.13) \qquad u_{\rho_i}(x) = e^{\rho_i \cdot x}\big(1 + O(t^{-1})\big), \quad x \in \overline{\Omega},$$

$$(7.14) \qquad \nabla u_{\rho_\ell}(x) = e^{\rho_i \cdot x}\big(\rho_i + O(1)\big), \quad x \in \overline{\Omega},$$

where the O symbols hide constants that are independent of x and t.

We start with the first constraint (7.12a). We have $|u_{\rho_1}(x)| = e^{\frac{1}{2}tx_1}|1 + O(t^{-1})|$ by (7.13), and choosing t big enough yields

$$(7.15) \qquad \big|u_{\rho_1}(x)\big| \geq \tfrac{1}{2}e^{\frac{1}{2}tx_1} > 0, \quad x \in \overline{\Omega}.$$

Similarly, turning to the second constraint (7.12b), using (7.14) we obtain that for every $x \in \overline{\Omega}$

$$\det[\nabla u_{\rho_2} \cdots \nabla u_{\rho_{d+1}}](x) = e^{(\rho_2 + \cdots + \rho_{d+1}) \cdot x}\left(\det[\,\rho_2 \,\cdots\, \rho_{d+1}\,] + O(t^{d-1})\right).$$

A direct calculation shows that $\det[\,\rho_2 \,\cdots\, \rho_{d+1}\,] = t^d(1-(-i)^d)$, and as a result

$$\det\left[\nabla u_{\rho_2} \cdots \nabla u_{\rho_{d+1}}\right](x) = t^d e^{(\rho_2 + \cdots + \rho_{d+1}) \cdot x}\left(1 - (-i)^d + O(t^{-1})\right).$$

Thus, since $|1 - (-i)^d| \geq \sqrt{2}$, choosing t big enough yields

(7.16) $$\left|\det\left[\nabla u_{\rho_2} \,\cdots\, \nabla u_{\rho_{d+1}}\right](x)\right| > 0, \quad x \in \overline{\Omega}.$$

We now consider constraint (7.12c). Using (7.13) and (7.14), we readily compute for $x \in \overline{\Omega}$

$$\det\begin{bmatrix} u_{\rho_1} & \cdots & u_{\rho_{d+1}} \\ \nabla u_{\rho_1} & \cdots & \nabla u_{\rho_{d+1}} \end{bmatrix}(x) = e^{(\rho_1 + \cdots + \rho_{d+1}) \cdot x}\det\begin{bmatrix} 1+O(t^{-1}) & \cdots & 1+O(t^{-1}) \\ \rho_1 + O(1) & \cdots & \rho_{d+1} + O(1) \end{bmatrix}$$
$$= t^d e^{(\rho_1 + \cdots + \rho_{d+1}) \cdot x}\left(\det\begin{bmatrix} 1 & \cdots & 1 \\ \rho_1/t & \cdots & \rho_{d+1}/t \end{bmatrix} + O(t^{-1})\right).$$

Using that $\rho_2 = 2\rho_1$ and subtracting twice the first column to the second column of $\begin{bmatrix} 1 & \cdots & 1 \\ \rho_1/t & \cdots & \rho_{d+1}/t \end{bmatrix}$ we have

$$\det\begin{bmatrix} 1 & -1 & \cdots & 1 \\ \rho_1/t & 0 & \cdots & \rho_{d+1}/t \end{bmatrix} = \tfrac{1}{2}t^{-d}\det[\rho_1\,\rho_3\,\cdots\,\rho_{d+1}]$$
$$= \tfrac{1}{2}t^{-d}\det[\,\rho_2\,\cdots\,\rho_{d+1}\,] = \tfrac{1}{2}\left(1-(-i)^d\right).$$

And again choosing t big enough yields

(7.17) $$\left|\det\begin{bmatrix} u_{\rho_1} & \cdots & u_{\rho_{d+1}} \\ \nabla u_{\rho_1} & \cdots & \nabla u_{\rho_{d+1}} \end{bmatrix}(x)\right| > 0, \quad x \in \overline{\Omega}.$$

In view of (7.15), (7.16) and (7.17) the constraints in (7.12) are verified for every $x \in \overline{\Omega}$ setting $\varphi_i = u_{\rho_i|\partial\Omega}$ for every i, since by (7.11) this implies $u^i = u_{\rho_i}$ in Ω. A standard elliptic regularity theory argument (as in the proof of Theorem 7.2) ensures that (7.12) still holds true for an open set of boundary values in $C^2(\overline{\Omega};\mathbb{C})^{d+1}$ near $(\varphi_i = u_{\rho_i|\partial\Omega})_i$. $\qquad\square$

7.3. The Runge approximation property

7.3.1. The main result. — Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain. We consider the elliptic boundary value problem [3]

$$(7.18) \qquad \begin{cases} Lu := -\operatorname{div}(a\nabla u) + qu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $q \in L^\infty(\Omega; \mathbb{R})$ and $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ satisfy

$$(7.19a) \qquad \begin{cases} \Lambda^{-1}|\xi|^2 \le a\xi \cdot \xi \le \Lambda|\xi|^2, & \xi \in \mathbb{R}^d, \\ |q| \le \Lambda, & \text{almost everywhere in } \Omega \end{cases}$$

for some $\Lambda > 0$ and

$$(7.19b) \qquad\qquad\qquad {}^Ta = a \quad \text{in } \Omega.$$

We assume that the problem is well-posed, that is,

$$(7.19c) \qquad 0 \text{ is not an eigenvalue for the operator } L \text{ in } H_0^1(\Omega; \mathbb{R}).$$

We start with the definition of the Runge approximation property [**136**].

Definition 7.4. — *We say that L satisfies the Runge approximation property if for any Lipschitz simply connected domain $\Omega_1 \Subset \Omega$ and any $u \in H^1(\Omega_1; \mathbb{R})$ such that $Lu = 0$ in Ω_1 there exists a sequence $u_n \in H^1(\Omega; \mathbb{R})$ such that*

1) $Lu_n = 0$ in Ω, and
2) $u_{n|\Omega_1} \to u$ in $L^2(\Omega_1; \mathbb{R})$.

The Runge approximation property holds true provided that the operator L satisfies the unique continuation property. The latter is a classical result in elliptic PDE theory.

Lemma 7.5 (Unique continuation property [21]). — *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz connected bounded domain, $\Sigma \subseteq \partial\Omega$ be an open non-empty portion of $\partial\Omega$, and a*

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3. This is not specific to this model: any elliptic PDE with complex coefficients could be considered. We restrict ourselves to this simpler case to avoid technicalities. For the general case, the reader is referred to [**136**], [**51**].

in $L^\infty(\Omega; \mathbb{R}^{d \times d})$ and $q \in L^\infty(\Omega; \mathbb{R})$ satisfy (7.19). If $d \geq 3$, assume that a is Lipschitz continuous. Let $u \in H^1(\Omega; \mathbb{R})$ be a solution to $Lu = 0$ in Ω. If

$$u = 0 \quad \text{and} \quad a\nabla u \cdot \nu = 0 \text{ on } \Sigma,$$

then $u = 0$ in Ω.

Remark 7.6. — In the particular case where a is isotropic, when $d \geq 3$ the assumption on the Lipschitz continuity of a may be reduced to $a \in W^{1,d}(\Omega; \mathbb{R})$ [**213**], in view of the equivalence of the uniqueness of the Cauchy problem with the weak unique continuation property [**173**].

The proof of this result goes beyond the scopes of this book. We now verify that the Runge approximation property follows from the unique continuation property for the model we consider.

Theorem 7.7 (Runge approximation). — Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain, $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ and $q \in L^\infty(\Omega; \mathbb{R})$ satisfy (7.19). If $d \geq 3$, assume that a is Lipschitz continuous. Then L defined in (7.18) satisfies the Runge approximation property.

Proof. — Without loss of generality, assume that Ω is connected. Take $\Omega_1 \Subset \Omega$ as in Definition 7.4 and $u \in H^1(\Omega_1; \mathbb{R})$ such that

(7.20) $$Lu = 0 \quad \text{in } \Omega_1.$$

Set $F = \{v_{|\Omega_1} : v \in H^1(\Omega; \mathbb{R}), \ Lv = 0 \text{ in } \Omega\}$. Suppose by contradiction that the Runge approximation property does not hold. By the Hahn–Banach theorem, there exists a functional $g \in L^2(\Omega_1; \mathbb{R})^*$ such that $g(u) \neq 0$ and $g(v) = 0$ for all $v \in F$. In other words, there exists $g \in L^2(\Omega_1; \mathbb{R})$ such that $(g, u)_{L^2(\Omega_1)} \neq 0$ and $(g, v)_{L^2(\Omega_1)} = 0$ for all $v \in F$.

Consider now the extension by zero of g to Ω, which by an abuse of notation is still denoted by g. Let $w \in H^1(\Omega; \mathbb{R})$ be the unique solution to

$$\begin{cases} Lw = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix now $\varphi \in H^{1/2}(\partial\Omega; \mathbb{R})$ and let $v \in H^1(\Omega; \mathbb{R})$ be the unique solution to

$$\begin{cases} Lv = 0 & \text{in } \Omega, \\ v = \varphi & \text{on } \partial\Omega. \end{cases}$$

By definition of g there holds $(g, v)_{L^2(\Omega)} = 0$. Thus, integration by parts shows that

$$0 = -(v, g)_{L^2(\Omega)} = (Lv, w)_{L^2(\Omega)} - (v, Lw)_{L^2(\Omega)} = \int_{\partial\Omega} (a\nabla w \cdot \nu)\varphi \, d\sigma.$$

Since the above identity holds for all $\varphi \in H^{1/2}(\partial\Omega; \mathbb{R})$, we obtain $a\nabla w \cdot \nu = 0$ on $\partial\Omega$. Observe now that w is solution to $Lw = 0$ in $\Omega \setminus \Omega_1$ such that $w = 0$ and $a\nabla w \cdot \nu = 0$ on $\partial\Omega$. In view of Lemma 7.5 we have $w = 0$ in $\Omega \setminus \Omega_1$, therefore $w = 0$ and $a\nabla w \cdot \nu = 0$ on $\partial\Omega_1$. As a result, by integrating by parts we obtain

$$\int_{\Omega_1} gu \, dx = \int_{\Omega_1} (Lw)u \, dx$$

$$= \int_{\Omega_1} -\operatorname{div}(a\nabla w)u + quw \, dx$$

$$= \int_{\Omega_1} a\nabla w \cdot \nabla u + quw \, dx + \int_{\partial\Omega_1} ua\nabla w \cdot \nu \, d\sigma$$

$$= \int_{\Omega_1} -\operatorname{div}(a\nabla u)w + qwu \, dx + \int_{\partial\Omega_1} wa\nabla u \cdot \nu \, d\sigma$$

$$= \int_{\Omega_1} (Lu)w \, dx = 0,$$

where the last identity follows from (7.20). This contradicts the assumptions on g, since $(g, u)_{L^2(\Omega_1)} \neq 0$, and the proof is concluded. $\quad\square$

We have seen that under quite general regularity assumptions on the coefficients, the Runge approximation property always holds. As a consequence, any local solution to $Lu = 0$ can be approximated by restrictions of global solutions in the L^2 norm. However, in view of the applications to the non-zero constraints we are interested in, we shall need a stronger norm.

Definition 7.8. — Take $\alpha \in (0, 1)$. We say that L satisfies the strong Runge approximation property if for any smooth simply connected domains $\Omega_2 \Subset$

$\Omega_1 \Subset \Omega$ and any $u \in H^1(\Omega_1; \mathbb{R}) \cap C^{1,\alpha}(\Omega_1; \mathbb{R})$ such that $Lu = 0$ in Ω_1 there exists a sequence $u_n \in H^1(\Omega; \mathbb{R}) \cap C^{1,\alpha}(\Omega_1; \mathbb{R})$ such that

   1) $Lu_n = 0$ in Ω, and
   2) $u_{n|\Omega_2} \to u_{|\Omega_2}$ in $C^{1,\alpha}(\overline{\Omega_2}; \mathbb{R})$.

Under suitable regularity assumptions, the strong Runge approximation property is an immediate consequence of the Runge approximation property and of standard elliptic regularity.

Corollary 7.9. — *Assume that the hypotheses of Theorem 7.7 hold true. Take $\alpha \in (0, 1)$ and suppose that $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})$. Then L satisfies the strong Runge approximation property.*

Proof. — Take $\Omega_2 \Subset \Omega_1 \Subset \Omega$ and u as in Definition 7.8. In view of Theorem 7.7, L satisfies the Runge approximation property. Namely, there exists a sequence $u_n \in H^1(\Omega; \mathbb{R})$ of solutions to $Lu_n = 0$ in Ω such that $u_{n|\Omega_1} \to u$ in $L^2(\Omega_1; \mathbb{R})$. Since $L(u_n - u) = 0$ in Ω_1, standard elliptic regularity [**103**, Theorem 5.20] gives $u_n \in C^{1,\alpha}(\Omega_1; \mathbb{R})$ and

$$\|u_n - u\|_{C^{1,\alpha}(\overline{\Omega_2})} \le C \|u_n - u\|_{L^2(\Omega_1)} \to 0,$$

as desired. □

It is worth observing that, by classical elliptic regularity theory, the regularity assumptions on the coefficients are minimal.

Definition 7.8 and Corollary 7.9 easily extend to the higher regularity case. This would allow to consider constraints depending on higher derivatives of u.

7.3.2. Application of the Runge approximation to internal non-zero constraints enforced via boundary control. — This approach is based on approximating locally the solutions to the constant coefficient case by means of the (strong) Runge approximation property.

In order to do this, consider the constant coefficient differential operator defined by

$$L_{x_0} = -\operatorname{div}\left(a(x_0)\nabla \bullet\right)$$

for $x_0 \in \Omega$. Note that, being of lower-order, the term in q is unnecessary.

By using the (strong) Runge approximation property, it is possible to approximate local solutions to $L_{x_0} u_0 = 0$ in $B(x_0, r)$ for some small $r > 0$ with global solutions u to $Lu = 0$ in Ω.

Proposition 7.10. *— Assume that the hypotheses of Corollary 7.9 hold true and take $\delta > 0$, $\Omega' \Subset \Omega$ and $u_0 \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R})$. There exists $r > 0$ depending on Ω, Ω', α, Λ, $\|a\|_{C^{0,\alpha}(\overline{\Omega})}$, $\|u_0\|_{C^{1,\alpha}(\overline{\Omega})}$ and δ such that for any $x_0 \in \overline{\Omega'}$ if $L_{x_0} u_0 = 0$ in $B(x_0, r)$ then there exists $\varphi_{x_0,\delta} \in H^{1/2}(\partial\Omega; \mathbb{R})$ such that*

$$\|u_{x_0,\delta} - u_0\|_{C^1(\overline{B(x_0,r)})} \leq \delta,$$

where $u_{x_0,\delta}$ is the solution of

$$\begin{cases} L u_{x_0,\delta} = 0 & in\ \Omega, \\ u_{x_0,\delta} = \varphi_{x_0,\delta} & on\ \partial\Omega. \end{cases}$$

This result follows from the strong Runge approximation property and standard elliptic regularity results. More precisely, we first approximate u_0 in the ball $B(x_0, 2r)$ by a local solution u_1 to $Lu_1 = 0$, for r small enough. Then, u_1 can be locally approximated in $B(x_0, r)$ by a global solutions thanks to the strong Runge approximation. The details of the proof, although fairly simple, are quite technical, and so are presented in §7.3.3 below.

This result can be extended to the higher regularity case. The case $\Omega' = \Omega$ could be handled as well, but would require additional technicalities to deal with the case when $x_0 \in \partial\Omega$ [**51**].

We now apply this result to the boundary control of elliptic PDE to enforce non-zero constraints.

7.3.2.1. *The Jacobian of solutions to the conductivity equation.* — As a natural generalisation of Chapter 6 and §7.2.3.1, we first consider the conductivity equation in $d \geq 2$ dimensions with anisotropic coefficient. Consider problem

$$(7.21) \qquad \begin{cases} -\operatorname{div}(a\nabla u^i) = 0 & in\ \Omega, \\ u^i = \varphi_i & on\ \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is a Lipschitz bounded domain and $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{d\times d})$ satisfies (7.19a) and (7.19b), and look for solutions such that $|\det[\nabla u^1 \ \cdots \ \nabla u^d]| > 0$ at least locally in Ω. The main result reads as follows.

Theorem 7.11 (see [43]). — *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain, $\Omega' \Subset \Omega$ and suppose that $a \in C^{0,1}(\overline{\Omega}; \mathbb{R}^{d \times d})$ satisfy (7.19a) and (7.19b). Then there exist $N = N(\Omega, \Omega', \Lambda, \|a\|_{C^{0,1}(\overline{\Omega})}) \in \mathbb{N}^*$, $r = r(\Omega, \Omega', \Lambda, \|a\|_{C^{0,1}(\overline{\Omega})}) > 0$, $x_1, \ldots, x_N \in \overline{\Omega'}$ and $\varphi_i^j \in H^{1/2}(\partial \Omega; \mathbb{R})$, $i = 1, \ldots, d$, $j = 1, \ldots, N$ such that*

$$\overline{\Omega'} \subseteq \bigcup_{j=1}^N B(x_j, r) \quad and \quad \det\big[\nabla u_{(j)}^1 \; \cdots \; \nabla u_{(j)}^d\big] \geq \tfrac{1}{2} \quad in \;\; B(x_j, r),$$

where $u_{(j)}^i \in H^1(\Omega; \mathbb{R})$ is the unique solution to (7.21) with boundary condition φ_i^j.

Remark 7.12. — A comparison of Theorem 7.11 and Theorem 7.2, where CGO solutions were used, leads to the following observations:

▷ The regularity requirements in Theorem 7.11 are lower than in Theorem 7.2.

▷ The conductivity a in Theorem 7.11 is matrix-valued and not scalar valued.

▷ In Theorem 7.2, exactly four boundary conditions are used when $d = 3$, while this result requires $3N$ boundary values where N can be determined *a priori* but may not be small. In other words, here the constraint is not satisfied globally, but merely in small balls of fixed radius covering the subdomain $\overline{\Omega'}$.

This result is based on Proposition 7.10, which allows us to approximate solution of PDE with variable coefficients by solutions of PDE with constant coefficients: a wide variety of constraints can be tackled by the same approach.

Proof. — We consider the d solutions to the constant coefficient case defined by $u_0^i = x_i$, for $i = 1, \ldots, d$. These are solutions to the constant coefficient equation in any point of the domain, namely for any $x_0 \in \overline{\Omega'}$

$$L_{x_0} u_0^i = -\operatorname{div}\big(a(x_0)\nabla x_i\big) = 0.$$

Hence, by Proposition 7.10, for any $\delta > 0$ there exists $r_\delta > 0$ depending on Ω, Ω', Λ, $\|a\|_{C^{0,1}(\overline{\Omega})}$ and δ and $\varphi_{x_0,\delta}^i \in H^{1/2}(\partial\Omega; \mathbb{R})$ such that one has

$\|u^i_{x_0,\delta} - x_i\|_{C^1(\overline{B(x_0,r_\delta)})} \leq \delta$, where $u^i_{x_0,\delta}$ is defined by

$$\begin{cases} -\operatorname{div}(a\nabla u^i_{x_0,\delta}) = 0 & \text{in } \Omega, \\ u^i = \varphi^i_{x_0,\delta} & \text{on } \partial\Omega. \end{cases}$$

Then $\left\|\nabla u^i_{x_0,\delta} - \mathbf{e}_i\right\|_{C^0(\overline{B(x_0,r_\delta)})} \leq \delta$, and we obtain

$$\left|\det\left[\nabla u^1_{x_0,\delta} \ \cdots \ \nabla u^d_{x_0,\delta}\right] - \det\left[\mathbf{e}_1 \ \cdots \ \mathbf{e}_d\right]\right| \leq \tfrac{1}{2} \quad \text{in } B(x_0,r_\delta),$$

provided that δ is chosen small enough. As a result

$$(7.22) \qquad \left|\det\left[\nabla u^1_{x_0,\delta} \ \cdots \ \nabla u^d_{x_0,\delta}\right]\right| \geq \tfrac{1}{2} \quad \text{in } B(x_0,r_\delta).$$

Since $\overline{\Omega'} \subseteq \bigcup_{x_0\in\overline{\Omega'}} B(x_0,r_\delta)$, by compactness there exist $x_1,\ldots,x_N \in \overline{\Omega'}$ such that $\overline{\Omega'} \subseteq \bigcup_{j=1}^N B(x_j,r_\delta)$. Thanks to (7.22), choosing $\varphi^j_i = \varphi^i_{x_j,\delta}$ concludes the proof. $\qquad\square$

7.3.2.2. *Application to the Schrödinger equation.* — We apply here the Runge approximation to the problem considered in §7.2.3.2. More precisely, let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain for some $d \geq 2$ and consider

$$(7.23) \qquad \begin{cases} -\operatorname{div}(a\nabla u^i) + qu^i = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $a \in C^{0,1}(\overline{\Omega};\mathbb{R}^{d\times d})$ and $q \in L^\infty(\Omega;\mathbb{R})$ satisfy (7.19).

We look for solutions u^i satisfying the constraints given in (7.12). The Runge approximation approach allows to satisfy these conditions locally in the interior of the domain Ω. The main result, which can be found in [**51**] in a different form, reads as follows.

Theorem 7.13. — *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz bounded domain for some $d \geq 2$, $\Omega' \Subset \Omega$ and suppose that $a \in C^{0,1}(\overline{\Omega};\mathbb{R}^{d\times d})$ and $q \in L^\infty(\Omega;\mathbb{R})$ satisfy (7.19). Then there exist*

$$N = N(\Omega,\Omega',\Lambda,\|a\|_{C^{0,1}(\overline{\Omega})}) \in \mathbb{N}^*, \quad r = r(\Omega,\Omega',\Lambda,\|a\|_{C^{0,1}(\overline{\Omega})}) > 0,$$

$$x_1,\ldots,x_N \in \overline{\Omega'}, \quad \varphi^j_i \in H^{1/2}(\partial\Omega;\mathbb{R}), \quad i = 1,\ldots,d+1, \, j = 1,\ldots,N,$$

such that

$$\overline{\Omega'} \subseteq \bigcup_{j=1}^{N} B(x_j, r)$$

and for every $j = 1, \ldots, N$ *and* $x \in B(x_j, r)$ *we have*

(7.24a)
$$\left| u_{(j)}^1(x) \right| \geq \tfrac{1}{2},$$

(7.24b)
$$\left| \det \left[\nabla u_{(j)}^2 \ \cdots \ \nabla u_{(j)}^{d+1} \right](x) \right| \geq \tfrac{1}{2},$$

(7.24c)
$$\left| \det \begin{bmatrix} u_{(j)}^1 & \cdots & u_{(j)}^{d+1} \\ \nabla u_{(j)}^1 & \cdots & \nabla u_{(j)}^{d+1} \end{bmatrix}(x) \right| \geq \tfrac{1}{2},$$

where $u_{(j)}^i \in H^1(\Omega; \mathbb{R})$ *is the unique solution to* (7.23) *with boundary condition* φ_i^j.

Compared to Theorem 7.3, Theorem 7.11 presents the advantages and shortcomings described in Remark 7.12.

Proof. — The proof of this theorem is based on Proposition 7.10 and follows the same strategy of the proof of Theorem 7.11. For $x^0 \in \overline{\Omega'}$, we consider problem (7.23) with the leading order coefficient frozen in x^0, and without the zero-th order term, namely

$$L_{x_0} = -\operatorname{div}\left(a(x_0)\nabla \bullet\right).$$

Consider now the $d+1$ solutions to this PDE defined by $u_0^1 = 1$ and $u_0^i = x_{i-1}$, for $i = 2, \ldots, d+1$. These are solutions to the constant coefficient equation in the whole domain, namely $L_{x_0} u_0^i = 0$. Hence, by Proposition 7.10, for any $\delta > 0$ there exists $r_\delta > 0$ depending on Ω, Ω', Λ, $\|a\|_{C^{0,1}(\overline{\Omega})}$ and δ and $\varphi_{x_0,\delta}^i \in H^{1/2}(\partial\Omega; \mathbb{R})$ such that

$$\left\| u_{x_0,\delta}^i - u_0^i \right\|_{C^1(\overline{B(x_0,r_\delta)})} \leq \delta,$$

where $u_{x_0,\delta}^i$ is defined by

$$\begin{cases} -\operatorname{div}(a\nabla u_{x_0,\delta}^i) + q u_{x_0,\delta}^i = 0 & \text{in } \Omega, \\ u^i = \varphi_{x_0,\delta}^i & \text{on } \partial\Omega. \end{cases}$$

In particular, we have for every $i = 2, \ldots, d+1$

$$\left\| u_{x_0,\delta}^1 - 1 \right\|_{C^0(\overline{B(x_0,r_\delta)})} \leq \delta, \ \left\| \nabla u_{x_0,\delta}^1 \right\|_{C^0(\overline{B(x_0,r_\delta)})} \leq \delta, \ \left\| \nabla u_{x_0,\delta}^i - \mathbf{e}_i \right\|_{C^0(\overline{B(x_0,r_\delta)})} \leq \delta,$$

thus for every $x \in B(x_0, r_\delta)$

$$\left| u^1_{x_0,\delta}(x) - 1 \right| \leq \tfrac{1}{2},$$

$$\left| \det \left[\nabla u^2_{x_0,\delta} \cdots \nabla u^{d+1}_{x_0,\delta} \right] (x) - \det \left[\mathbf{e}_1 \cdots \mathbf{e}_d \right] \right| \leq \tfrac{1}{2},$$

$$\left| \det \begin{bmatrix} u^1_{x_0,\delta} & \cdots & u^{d+1}_{x_0,\delta} \\ \nabla u^1_{x_0,\delta} & \cdots & \nabla u^{d+1}_{x_0,\delta} \end{bmatrix} (x) - \det \begin{bmatrix} 1 & u^2_{x_0,\delta} & \cdots & u^{d+1}_{x_0,\delta} \\ 0 & \mathbf{e}_1 & \cdots & \mathbf{e}_d \end{bmatrix} \right| \leq \tfrac{1}{2},$$

provided that δ is chosen small enough. As a result, for every $x \in B(x_0, r_\delta)$

$$\left| u^1_{x_0,\delta} \right| \geq \tfrac{1}{2},$$

(7.25)
$$\left| \det \left[\nabla u^2_{x_0,\delta} \cdots \nabla u^{d+1}_{x_0,\delta} \right] (x) \right| \geq \tfrac{1}{2},$$

$$\left| \det \begin{bmatrix} u^1_{x_0,\delta} & \cdots & u^{d+1}_{x_0,\delta} \\ \nabla u^1_{x_0,\delta} & \cdots & \nabla u^{d+1}_{x_0,\delta} \end{bmatrix} (x) \right| \geq \tfrac{1}{2}.$$

Since $\overline{\Omega'} \subseteq \bigcup_{x_0 \in \overline{\Omega'}} B(x_0, r_\delta)$, by compactness there exist $x_1, \ldots, x_N \in \overline{\Omega'}$ such that $\overline{\Omega'} \subseteq \bigcup_{j=1}^{N} B(x_j, r_\delta)$. We conclude the proof by choosing $\varphi^j_i = \varphi^i_{x_j,\delta}$ thanks to (7.25). $\qquad\square$

7.3.3. Proof of Proposition 7.10. — The proof of Proposition 7.10 is based on the strong Runge approximation property and on the elliptic regularity theory. We prove below the regularity estimate we need. The result is classical, but the proof is given to show that the relevant constant does not depend on the size of the domain.

Lemma 7.14. — *Take $s \in (0, 1]$, $\alpha \in (0, 1)$ and $x_0 \in \mathbb{R}^d$. Take*

$$q \in L^\infty(B(x_0, s); \mathbb{R}), \qquad a \in C^{0,\alpha}(\overline{B(x_0, s)}; \mathbb{R}^{d \times d}) \text{ such that (7.19a) holds,}$$

$$F \in C^{0,\alpha}(\overline{B(x_0, s)}; \mathbb{R}^d), \qquad f \in L^{\frac{d}{1-\alpha}}(B(x_0, s); \mathbb{R}).$$

Let $u \in H^1(B(x_0, s); \mathbb{R})$ be the unique solution to

$$\begin{cases} (Lu =) - \operatorname{div}(a \nabla u) + qu = \operatorname{div} F + f & \text{in } B(x_0, s), \\ u = 0 & \text{on } \partial B(x_0, s). \end{cases}$$

Then $u \in C^{1,\alpha}(\overline{B(x_0, s)}; \mathbb{R})$ *and*

$$\|u\|_{C^{1,\alpha}(\overline{B(x_0,s)})} \leq C\left(\|F\|_{C^{0,\alpha}(\overline{B(x_0,s)})} + \|f\|_{L^{d/(1-\alpha)}(B(x_0,s))}\right)$$

for some $C > 0$ *depending on* α, Λ *and* $\|a\|_{C^{0,\alpha}(\overline{B(x_0,s)})}$.

Proof. — Without loss of generality we set $x_0 = 0$. In order to obtain the independence of C of the radius s, we transform the problem in $B_s := B(0, s)$ into a problem defined in the unit ball B_1. Consider the map $\gamma_s \colon B_1 \to B_s$, $x \mapsto sx$. Given a function $g \in C^{0,\alpha}(B_s)$, the Hölder semi-norm of $g \circ \gamma_s$ can be written in terms of the semi-norm of g as follows:

$$(7.26) \qquad |g \circ \gamma_s|_{C^{0,\alpha}(B_1)} := \sup_{\substack{x,y \in B_1 \\ x \neq y}} \frac{|g(\gamma_s(x)) - g(\gamma_s(y))|}{|x - y|^\alpha}$$

$$= \sup_{\substack{x,y \in B_1 \\ x \neq y}} \frac{|g(\gamma_s(x)) - g(\gamma_s(y))|}{|\gamma_s(x) - \gamma_s(y)|^\alpha} \cdot \frac{|sx - sy|^\alpha}{|x - y|^\alpha}$$

$$= s^\alpha \sup_{\substack{x,y \in B_s \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|^\alpha} = s^\alpha |g|_{C^{0,\alpha}(B_s)}.$$

Similarly, if $g \in C^{1,\alpha}(B_s)$ there holds

$$(7.27) \qquad \|g \circ \gamma_s\|_{C^{1,\alpha}(\overline{B_1})}$$

$$= \|g \circ \gamma_s\|_{C^0(\overline{B_1})} + \|\nabla(g \circ \gamma_s)\|_{C^0(\overline{B_1})} + |\nabla(g \circ \gamma_s)|_{C^{0,\alpha}(B_1)}$$

$$= \|g\|_{C^0(\overline{B_s})} + s\|\nabla g \circ \gamma_s\|_{C^0(\overline{B_1})} + s|\nabla g \circ \gamma_s|_{C^{0,\alpha}(B_1)}$$

$$= \|g\|_{C^0(\overline{B_s})} + s\|\nabla g\|_{C^0(\overline{B_s})} + s^{1+\alpha}|\nabla g|_{C^{0,\alpha}(B_s)}$$

$$\geq s^{1+\alpha}\|g\|_{C^{1,\alpha}(\overline{B_s})}.$$

Consider now $v = u \circ \gamma_s$. A straightforward computation shows that v is the solution to

$$\begin{cases} -\operatorname{div}\left((a \circ \gamma_s)\nabla v\right) + s^2(q \circ \gamma_s)v = s \operatorname{div}\left(F \circ \gamma_s - F(0)\right) + s^2(f \circ \gamma_s) & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases}$$

Standard Schauder estimates for elliptic equations (see Corollary 8.35 and the following remark in [**104**]) applied to this problem give that $v \in C^{1,\alpha}(\overline{B_1})$ and

$$\|v\|_{C^{1,\alpha}(\overline{B_1})} \leq C\left(s\|F \circ \gamma_s - F(0)\|_{C^{0,\alpha}(\overline{B_1})} + s^2\|f \circ \gamma_s\|_{L^{d/(1-\alpha)}(B_1)}\right)$$

for some $C > 0$ depending on α, Λ and $\|a\|_{C^{0,\alpha}(\overline{B(x_0,s)})}$. Therefore, by (7.27) there holds

$$\|u\|_{C^{1,\alpha}(\overline{B_s})} \leq C\left(s^{-\alpha}\|F \circ \gamma_s - F(0)\|_{C^{0,\alpha}(\overline{B_1})} + \|f\|_{L^{d/(1-\alpha)}(B_s)}\right),$$

where we have also used the identity $\|f \circ \gamma_s\|_{L^{d/(1-\alpha)}(B_1)} = s^{\alpha-1}\|f\|_{L^{d/(1-\alpha)}(B_s)}$. Now note that in view of (7.26) we have

$$\|F \circ \gamma_s - F(0)\|_{C^{0,\alpha}(\overline{B_1})} = \|F \circ \gamma_s - F(0)\|_{C^0(\overline{B_1})} + |F \circ \gamma_s|_{C^{0,\alpha}(B_1)}$$

$$= \sup_{x \in B_1} \frac{|F(\gamma_s(x)) - F(\gamma_s(0))|}{|x|^\alpha}|x|^\alpha + |F \circ \gamma_s|_{C^{0,\alpha}(B_1)}$$

$$\leq 2|F \circ \gamma_s|_{C^{0,\alpha}(B_1)} = 2s^\alpha|F|_{C^{0,\alpha}(B_s)}.$$

Combining the last two inequalities we obtain

$$\|u\|_{C^{1,\alpha}(\overline{B_s})} \leq C\left(|F|_{C^{0,\alpha}(B_s)} + \|f\|_{L^{d/(1-\alpha)}(B_s)}\right)$$

for some $C > 0$ depending on α, Λ and $\|a\|_{C^{0,\alpha}(\overline{B(x_0,s)})}$, as desired. □

We are now in a position to prove Proposition 7.10.

Proof of Proposition 7.10. — Several positive constants depending on Ω, Ω', α, Λ, $\|a\|_{C^{0,\alpha}(\overline{\Omega})}$ and $\|u_0\|_{C^{1,\alpha}(\overline{\Omega})}$ will be denoted by the same letter C. During the proof, we shall need the following inequality:

Given $g \in C^{0,\alpha}(\overline{B(x_0,s)})$ for some $s \in (0,1]$ we have

(7.28) $\|g - g(x_0)\|_{C^{0,\alpha/2}(\overline{B(x_0,s)})}$

$$= \|g - g(x_0)\|_{C^0(\overline{B(x_0,s)})} + \sup_{\substack{x,y \in B(x_0,s) \\ x \neq y}} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \cdot |x - y|^{\frac{1}{2}\alpha}$$

$$\leq \sup_{x \in B(x_0,s)} \frac{|g(x) - g(x_0)|}{|x - x_0|^\alpha} \cdot |x - x_0|^\alpha + (2s)^{\frac{1}{2}\alpha}|g|_{C^{0,\alpha}(B(x_0,s))}$$

$$\leq \widetilde{C} s^{\frac{1}{2}\alpha}|g|_{C^{0,\alpha}(B(x_0,s))} \quad \text{for some absolute constant } \widetilde{C} > 0.$$

The proof is split into two steps. In the first step, we approximate u_0 with local solutions to the non-constant coefficient PDE. In the second step, we approximate these solutions with global solutions using the Runge approximation property.

Step 1. — Given $x_0 \in \overline{\Omega}'$ and for $s \in (0, \min(\mathrm{dist}(\partial\Omega, \Omega'), 1))$ suppose that $L_{x_0} u_0 = 0$ in $B(x_0, s)$. Let $u_s \in H^1(B(x_0, s))$ be the solution to the following problem

$$\begin{cases} L u_s = 0 & \text{in } B(x_0, s), \\ u_s = u_0 & \text{on } \partial B(x_0, s). \end{cases}$$

Let us show that $u_s \to u_0$ in a suitable Hölder norm as $s \to 0$. Consider the difference $v_s = u_s - u_0$, that is, the unique solution to the problem

$$\begin{cases} L v_s = -\mathrm{div}\left((a - a(x_0))\nabla u_0\right) + q u_0 & \text{in } B(x_0, s), \\ v_s = 0 & \text{on } \partial B(x_0, s). \end{cases}$$

In view of Lemma 7.14 there holds

$$\|v_s\|_{C^{1,\alpha/2}(\overline{B(x_0,s)})} \leq C\left(\left\|(a - a(x_0))\nabla u_0\right\|_{C^{0,\alpha/2}(\overline{B(x_0,s)})} + \|q u_0\|_{L^{d/(1-\alpha/2)}(B(x_0,s))}\right).$$

Let us analyse the first factor on the right-hand side: in view of (7.28) there holds

$$\left\|(a - a(x_0))\nabla u_0\right\|_{C^{0,\alpha/2}(\overline{B(x_0,s)})} \leq C\|a - a(x_0)\|_{C^{0,\alpha/2}(\overline{B(x_0,s)})} \cdot \|\nabla u_0\|_{C^{0,\alpha/2}(\overline{B(x_0,s)})}$$
$$\leq C s^{\alpha/2} |a|_{C^{0,\alpha}(\overline{B(x_0,s)})} \leq C s^{\alpha/2}.$$

Similarly we have

$$\|q u_0\|_{L^{d/(1-\alpha/2)}(B(x_0,s))} \leq C\|1\|_{L^{d/(1-\alpha/2)}(B(x_0,s))} \leq C s^{1-\alpha/2}.$$

Combining the last three inequalities we obtain

$$\|u_s - u_0\|_{C^{1,\alpha/2}(\overline{B(x_0,s)})} \leq C s^{\alpha/2}.$$

Hence there exists $\widetilde{s} > 0$ depending on $\Omega, \Omega', \alpha, \Lambda, \|a\|_{C^{0,\alpha}(\overline{B(x_0,s)})}, \|u_0\|_{C^{1,\alpha}(\overline{\Omega})}$ and δ such that

(7.29) $\|u_{\widetilde{s}} - u_0\|_{C^{1,\alpha/2}(\overline{B(x_0,\widetilde{s})})} \leq \tfrac{1}{2}\delta.$

Step 2. — By Corollary 7.9, L satisfies the strong Runge approximation property, which we apply to $u_{\widetilde{s}}$ with $\Omega_2 = B(x_0, \frac{1}{2}\widetilde{s})$ and $\Omega_1 = B(x_0, \widetilde{s})$ (see Definition 7.8). There exists a sequence $u_n \in H^1(\Omega; \mathbb{R}) \cap C^{1,\alpha}(B(x_0, \widetilde{s}); \mathbb{R})$ such that $Lu_n = 0$ in Ω and

$$\|u_n - u_{\widetilde{s}}\|_{C^{1,\alpha}(\overline{B(x_0, \widetilde{s}/2)})} \xrightarrow[n]{} 0.$$

As a consequence, there exists n such that

$$\|u_n - u_{\widetilde{s}}\|_{C^{1,\alpha}(\overline{B(x_0, \widetilde{s}/2)})} \leq \tfrac{1}{2}\delta.$$

Hence, by (7.29) we obtain

$$\|u_n - u_0\|_{C^1(\overline{B(x_0, \widetilde{s}/2)})} \leq \delta.$$

Setting $r = \frac{1}{2}\widetilde{s}$ and $\varphi_{x_0, \delta} = u_n|_{\partial\Omega}$ concludes the proof. $\qquad\square$

CHAPTER 8

USING MULTIPLE FREQUENCIES TO ENFORCE NON-ZERO CONSTRAINTS ON SOLUTIONS OF BOUNDARY VALUE PROBLEMS

8.1. Introduction

In Chapter 6 and Chapter 7 we reviewed several techniques designed to ensure that the solutions of boundary value problems satisfy prescribed interior local non-zero constraints. In Chapter 6, we considered the conductivity equation

$$- \operatorname{div}(a \nabla u^i) = 0 \quad \text{in } \Omega,$$

and the Jacobian constraint

$$\left| \det \left[\nabla u^1 \ \cdots \ \nabla u^d \right] \right| \geq C > 0.$$

We showed that, if $d = 2$, it is possible to enforce the above condition for any a simply by choosing the boundary values x_1 and x_2, provided that Ω is convex. This method makes strong use of the fact that we are in two dimensions and of the maximum principle. It cannot be generalised to higher dimensions (see Section 6.5) or to Helmholtz-type equations.

In Chapter 7, we presented two methods that can be used to overcome these issues, the complex geometric optics solutions and the Runge approximation property. These approaches can successfully be used in any dimension with more general problems of the type

$$\operatorname{div}(a \nabla u^i) + q u^i = 0 \quad \text{in } \Omega,$$

and for several types of constraints. However, they have a common drawback: the suitable boundary conditions may not be constructed *a priori*, independently of the coefficients. This is clearly a serious issue in inverse problems, where the parameters of the PDE are unknown.

This chapter focuses on a different approach to this problem based on the use of multiple frequencies. As such, this method is applicable only with frequency dependent problems. We consider the second-order elliptic PDE

$$(8.1) \qquad \begin{cases} \operatorname{div}(a\nabla u_\omega^i) + (\omega^2\varepsilon + \mathrm{i}\omega\sigma)u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $a \in C^{0,\alpha}(\overline{\Omega};\mathbb{R}^{d\times d})$ is a uniformly elliptic tensor and $\varepsilon, \sigma \in L^\infty(\Omega;\mathbb{R}_+)$. The case $\sigma \equiv 0$ could be considered as well [3], [5], but the presence of real eigenvalues makes the analysis slightly more involved: in this book we have decided to deal only with the simpler case $\sigma > 0$. This Helmholtz-type equation is a scalar approximation of Maxwell's system, with a being the inverse of the magnetic permeability, ε the electric permittivity and σ the electric conductivity.

In addition to the Jacobian constraint

$$(8.2) \qquad \left| \det\left[\nabla u_\omega^2 \cdots \nabla u_\omega^{d+1} \right] \right| \geq C > 0$$

discussed in the previous chapters, we consider here the two conditions

$$(8.3) \qquad |u_\omega^1| \geq C > 0, \quad \left| \det\begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix} \right| \geq C > 0.$$

These constraints are motivated by the reconstruction methods of some hybrid imaging inverse problems discussed in Part II. They previously appeared in Chapter 7.

The key of this method is the availability of multiple frequencies in an admissible range $\mathcal{A} = [K_{\min}, K_{\max}]$, for some $0 < K_{\min} < K_{\max}$. In other words, we assume that we have access to measurements at several frequencies in a fixed range. For example, in thermoacoustic tomography the set $\mathcal{A}$ denotes the microwave range for electromagnetic waves. The reason why allowing for several frequencies makes enforcing the above constraints simpler is very intuitive: the zero level sets related to the constraints move when the frequency changes, provided that the boundary conditions are suitably chosen.
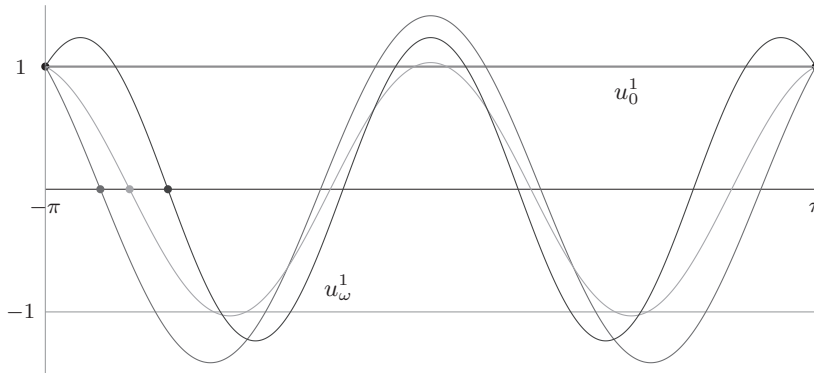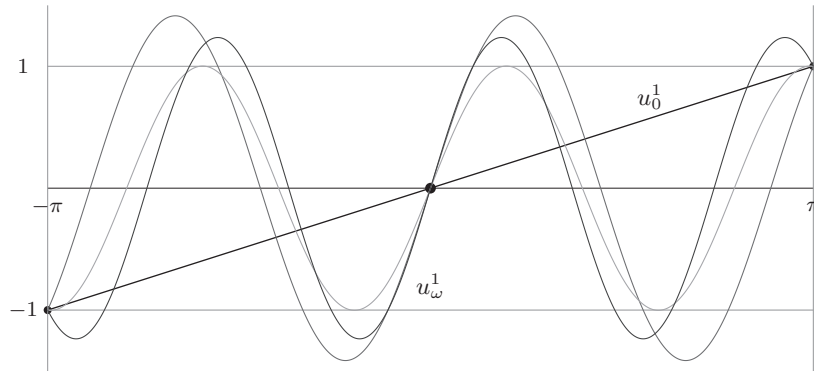
(a) $\varphi_1(-\pi) = 1$, $\varphi_1(\pi) = 1$

(b) $\varphi_1(-\pi) = -1$, $\varphi_1(\pi) = 1$

FIGURE 8.1. The multi-frequency approach in 1D. For a fixed boundary value φ_1, the solutions to (8.1) with $a \equiv \varepsilon \equiv 1$ and $\sigma \equiv 0$ are plotted for several frequencies ω.

It is instructive to consider the one dimensional case to visualise this phenomenon (see Figure 8.1). For simplicity, take $\Omega = (-\pi, \pi)$, $a \equiv \varepsilon \equiv 1$ and $\sigma \equiv 0$ and consider only the constraint $|u_\omega^1| > 0$. For a fixed boundary value φ_1 and a fixed frequency $\omega \in \mathcal{A}$, the corresponding solution u_ω^1 necessary cancels in Ω, provided that ω is bigger than the first Dirichlet eigenvalue. Fix now $\varphi_1(-\pi) = \varphi_1(\pi) = 1$ as in Figure 8.1 (A): the zeros move when the frequency

changes. In this case, it would be sufficient to choose φ_1 with two different frequencies in $\mathcal{A}$ in order to have the constraint satisfied everywhere for at least one solution. On the other hand, if the boundary value $-\varphi_1(-\pi) = \varphi_1(\pi) = 1$ is chosen as in Figure 8.1 (B), we have that $u_\omega^1(0) = 0$ for all ω. In other words, in $x = 0$ the constraint will never be satisfied, no matter how many frequencies are selected.

In order to understand why the first choice for φ_1 gives the desired behaviour while the second one does not, it is useful to look at the solution u_0^1 to (8.1) with the frequency ω set to nought. In the first case $(\varphi_1(-\pi) = \varphi_1(\pi) = 1$, Figure 8.1), we have $u_0^1(x) = 1$ for every $x \in \Omega$: u_0^1 satisfies the constraint $|u_\omega^1| > 0$ everywhere in Ω. On the other hand, in the second case $(-\varphi_1(-\pi) = \varphi_1(\pi) = 1$, Figure 8.1), we have $u_0^1(x) = x/\pi$ for every $x \in \Omega$: u_0^1 does not satisfy the constraint in $x = 0$. Thus, it seems that the behaviour of the zeros for positive frequencies depends on the zero-frequency case. More precisely, if the constraint is satisfied in $\omega = 0$ then the zeros should move when the frequency changes, as desired.

The reduction to the zero-frequency case allows us to simplify the problem substantially. Indeed, when $\omega = 0$, problem (8.1) becomes the conductivity equation

$$\begin{cases} -\operatorname{div}(a \nabla u_0^i) = 0 & \text{in } \Omega, \\ u_0^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

We can rely on Chapter 6 for guidance in this case. The constraint $|u_0^1| > 0$ can be easily satisfied in any dimension by choosing $\varphi_1 \equiv 1$, since this implies $u_0^1 \equiv 1$ (as in Figure 8.1). The Jacobian constraint (8.2) can be addressed in two dimensions thanks to the results of Chapter 6. In three dimensions, assuming that a is (close to) a constant matrix, it is enough to choose $\varphi_i \equiv x_{i-1}$ for $i = 2, 3, 4$, so that $u_0^i \equiv x_{i-1}$, and so $\det[\nabla u_0^2 \nabla u_0^3 \nabla u_0^4] \equiv 1$. Note that, also in three dimensions, we have no conditions on ε and σ, since they disappear from the PDE when $\omega = 0$. Finally, choosing $\varphi_1 \equiv 1$, the last constraint in (8.3) is an immediate consequence of the Jacobian condition.

Once the required constraints are satisfied in $\omega = 0$, it remains to show that these properties *transfer* to the range of frequencies $\mathcal{A}$. This is done quantitatively: the frequencies and the lower bound C are determined *a priori*, and

depend on the parameters of the PDE only through their *a priori* bounds. The proof is based on a quantitative version of the unique continuation theorem for holomorphic functions.

This method was introduced by the first author in a series of papers [**3**], [**6**], [**5**], where the extension to Maxwell's system is considered as well. Ammari et al. [**32**] generalised this technique to the conductivity equation with frequency-dependent complex coefficients; Robin boundary conditions were considered in [**10**]. See [**13**], [**7**] for other works on this subject.

This chapter is structured as follows. The main assumptions and results are discussed in Section 8.2, and the proofs are detailed in Section 8.3. An important tool is a quantitative unique continuation lemma for holomorphic functions, which is proved in Section 8.4.

8.2. Main results

Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,\alpha}$ bounded domain for some $\alpha \in (0,1)$ and with $d = 2$ or $d = 3$. We consider the Dirichlet boundary value problem

$$(8.4) \qquad \begin{cases} -\operatorname{div}(a\nabla u_\omega^i) - (\omega^2\varepsilon + i\omega\sigma)u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $a \in C^{0,\alpha}(\overline{\Omega};\mathbb{R}^{d\times d})$ satisfies

$$(8.5) \qquad \Lambda^{-1}|\xi|^2 \le a\xi \cdot \xi \le \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

and the regularity estimate

$$(8.6) \qquad \|a\|_{C^{0,\alpha}(\overline{\Omega})} \le \Lambda$$

for some $\Lambda > 0$ and $\varepsilon, \sigma \in L^\infty(\Omega;\mathbb{R})$ satisfy

$$(8.7) \qquad \Lambda^{-1} \le \varepsilon, \sigma \le \Lambda \quad \text{almost everyhere.}$$

According to Lemma 8.5, for $\varphi_i \in C^{1,\alpha}(\overline{\Omega};\mathbb{C})$ the above problem admits a unique solution $u_\omega^i \in H^1(\Omega;\mathbb{C})$. Moreover, by elliptic regularity theory, we have $u_\omega^i \in C^{1,\alpha}(\overline{\Omega};\mathbb{C})$. This property is fundamental for us: it allows us to take pointwise values of the solutions and of their gradients.

Let $\mathcal{A} = [K_{\min}, K_{\max}]$ be the admissible range of frequencies, for some $0 < K_{\min} < K_{\max}$. At the core of this method is the possibility of choosing
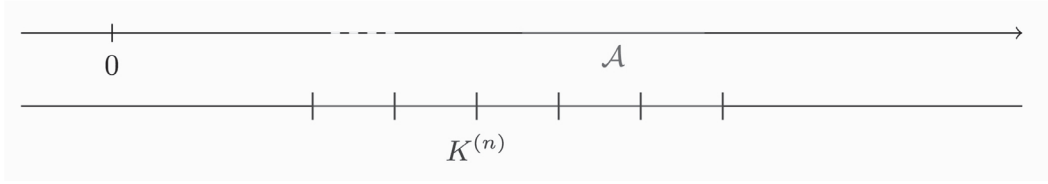
FIGURE 8.2. The admissible range $\mathcal{A}$ and the set of frequencies $K^{(n)}$.

multiple frequencies $\omega \in \mathcal{A}$. The easiest way to choose them is with a uniform sampling of $\mathcal{A}$. For $n \in \mathbb{N}$, $n \geq 2$, let $K^{(n)}$ be the uniform sampling of $\mathcal{A}$ of cardinality n, namely

$$(8.8) \qquad K^{(n)} = \left\{ K_{\min} + \frac{\ell - 1}{n - 1}(K_{\max} - K_{\min}) : \ell = 1, \ldots, n \right\},$$

(see Figure 8.2).

Let us state the main result of this chapter in the two-dimensional case. Thanks to the theory discussed in Chapter 6, there is no restriction on the leading order term a, other than (8.5) and (8.6).

Theorem 8.1. — *Let $\Omega \subseteq \mathbb{R}^2$ be a $C^{1,\alpha}$ bounded convex domain for some $\alpha \in (0,1)$ and take $\Omega' \Subset \Omega$. Assume that $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2 \times 2})$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy (8.5), (8.6) and (8.7) for some $\Lambda > 0$. Choose*

$$\varphi_1 = 1, \quad \varphi_2 = x_1 \quad and \quad \varphi_3 = x_2.$$

There exist $C > 0$ and $n \geq 2$ depending only on Ω, Ω', Λ, α and $\mathcal{A}$ and an open cover

$$\overline{\Omega'} = \bigcup_{\omega \in K^{(n)}} \Omega_\omega$$

such that for every $\omega \in K^{(n)}$ and $x \in \Omega_\omega$ we have

$$|u_\omega^1(x)| \geq C, \quad \left| \det \left[\nabla u_\omega^2 \, \nabla u_\omega^3 \right](x) \right| \geq C, \quad \left| \det \begin{bmatrix} u_\omega^1 & u_\omega^2 & u_\omega^3 \\ \nabla u_\omega^1 & \nabla u_\omega^2 & \nabla u_\omega^3 \end{bmatrix}(x) \right| \geq C,$$

where u_ω^i is given by (8.4).

The above result does not extend trivially to the three-dimensional case, in view of what we saw in Section 6.5. However, in the case when a is a constant matrix, the dimensionality restriction disappears.

Theorem 8.2. — *Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,\alpha}$ bounded domain for some $\alpha \in (0,1)$ and $d = 2$ or $d = 3$. Assume that $a \in \mathbb{R}^{d \times d}$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy (8.5), (8.6) and (8.7) for some $\Lambda > 0$. Choose*

$$\varphi_1 = 1, \quad \varphi_2 = x_1, \quad \dots \quad , \varphi_{d+1} = x_d.$$

There exist $C > 0$ and $n \geq 2$ depending only on Ω, Λ and $\mathcal{A}$ and an open cover

$$\overline{\Omega} = \bigcup_{\omega \in K^{(n)}} \Omega_\omega$$

such that for every $\omega \in K^{(n)}$ and $x \in \Omega_\omega$ we have

$$\left| u_\omega^1(x) \right| \geq C, \quad \left| \det \left[\nabla u_\omega^2 \; \cdots \; \nabla u_\omega^{d+1} \right](x) \right| \geq C, \quad \left| \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix}(x) \right| \geq C,$$

where u_ω^i is given by (8.4).

A simple example when $d = 2$ and $n = 2$ is showed in Figure 8.3. Some comments on these results are in order.

Remark 8.3. — In view of standard Schauder estimates, Theorem 8.2 holds true also in the case when a is a small $C^{0,\alpha}$ perturbation of a constant tensor. If we consider only the constraint $|u_\omega^1| \geq C$, this assumption can be removed, since the function $u_0^1 \equiv 1$ is always a solution to the zero-frequency PDE.

It remains an open question whether Theorem 8.2 holds true for any a if $d = 3$. In [**7**], it was proven that, under certain assumptions, for any a it is possible to satisfy the weaker constraint $|\nabla u_\omega| > 0$ by using multiple frequencies and a fixed generic boundary condition.

It is natural to wonder whether the above results hold true for any boundary conditions. The answer is no, as it can be seen in Figure 8.1 in the 1D case. More precisely, the odd boundary value φ_1 such that $(-1)\varphi_1(-\pi) = \varphi_1(\pi) = 1$, gives $u_\omega^1(0) = 0$ for every ω, and so the first constraint cannot be enforced with this boundary condition, no matter how many frequencies are selected. Similarly, the even boundary value φ_2 such
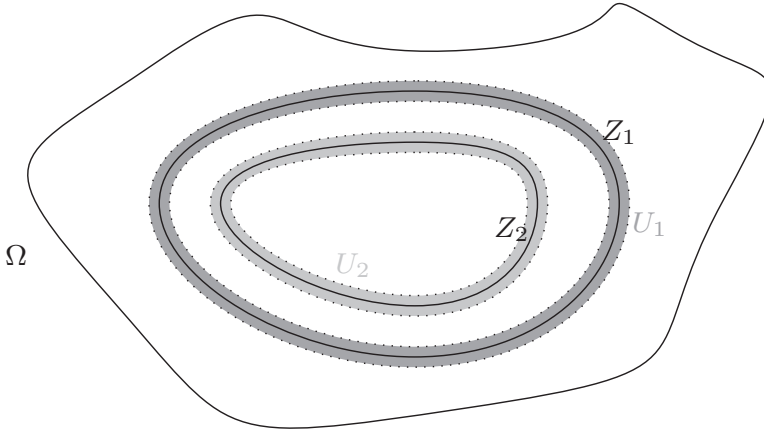
FIGURE 8.3. The multi-frequency approach when $d = 2$. Considering for simplicity only the constraint $|u_\omega^1(x)| \geq C$, let $Z_\ell = \{x \in \Omega : u_{\omega_\ell}^1(x) = 0\}$ be the corresponding nodal set and U_ℓ be a sufficiently small neighbourhood of Z_ℓ. Set $\Omega_{\omega_\ell} = \overline{\Omega} \setminus \overline{U_\ell}$, so that $|u_{\omega_\ell}^1| \geq C$ in Ω_{ω_ℓ} for some $C > 0$. We show here an example where two frequencies suffice, namely $\overline{\Omega} = \Omega_{\omega_1} \cup \Omega_{\omega_2}$. A more complicated example is shown in Figure 8.4.

that $\varphi_2(-\pi) = \varphi_2(\pi) = 1$, gives $\nabla u_\omega^2(0) = 0$ for every ω, and so the second constraint cannot be enforced.

Similar examples where the zero-level sets do not move when the frequency changes can be constructed in any dimension. For instance, as far as the first constraint is concerned, consider the case $d = 2$, $\Omega = B(0,1)$, $a \equiv \varepsilon \equiv 1$ and $\sigma \equiv 0$ and choose the boundary value $\varphi_1(x) = x_1$. In polar coordinates (ρ, θ), the corresponding solution is

$$u_\omega^1(\rho, \theta) = \frac{J_1(\omega\rho)}{J_1(\omega)} \cos \theta,$$

where J_1 is the Bessel function of the first kind of order 1. Therefore, u_ω^1 vanishes on the axis $\{x \in \Omega : x_1 = 0\}$ for every ω.

It is intuitive to see that these examples are pathological. Indeed, such choices of the boundary conditions exploit particular symmetries of the domain and of the coefficients. For generic boundary conditions, this pathological behaviour does not occur, and the multi-frequency method can be applied [**7**].

While the number of required frequencies n is in theory determined *a priori*, it would be desirable to have a reasonable estimates on how many frequencies are needed in practice. If the coefficients of the PDE are real analytic, then almost any choice of $d + 1$ frequencies in $\mathcal{A}^{d+1}$ gives the required constraints [**13**]. Examples of this result in dimension one and two can be seen in Figures 8.1 and 8.4, respectively. It follows from these examples that $d + 1$ is an optimal bound; namely, d frequencies may not suffice. However, in any dimension there are cases when two frequencies are sufficient, as in the situation depicted in Figure 8.3. An analytic example of this behaviour is given by the solutions

$$u_\omega(\rho, \theta) = \frac{J_0(\omega\rho)}{J_0(\omega)}, \quad \rho \geq 0, \ \theta \in [0, 2\pi),$$

to the constant coefficient case in $\Omega = B(0,1) \subseteq \mathbb{R}^2$ written in polar coordinates, with associated boundary value $\varphi \equiv 1$. Since these solutions are radial, the zero level sets of u_ω consist of circles around the origin that move when the frequency changes. Therefore, the nodal sets for different frequencies are not intersecting as in Figure 8.4, and two frequencies are sufficient.

Remark 8.4. — This method does not use the particular structure of this PDE or of the constraints considered here. Consequently the same approach works in the case $\sigma \equiv 0$, with Maxwell's equations or with other constraints, as long as these are satisfied for a particular frequency, *e.g.* $\omega = 0$ [**32**], [**5**], [**6**].

Before moving to the proofs of these results, let us compare them with those discussed in Chapter 7.

▷ The regularity assumptions on the parameters ($a \in C^{0,\alpha}$ and $\varepsilon, \sigma \in L^\infty$) are much lower than the assumptions needed for the CGO approach ($a \in C^3$ and $\varepsilon, \sigma \in C^1$) and the same as those related to the Runge approximation if $d = 2$. It is worth noting that the regularity assumed here is minimal if we
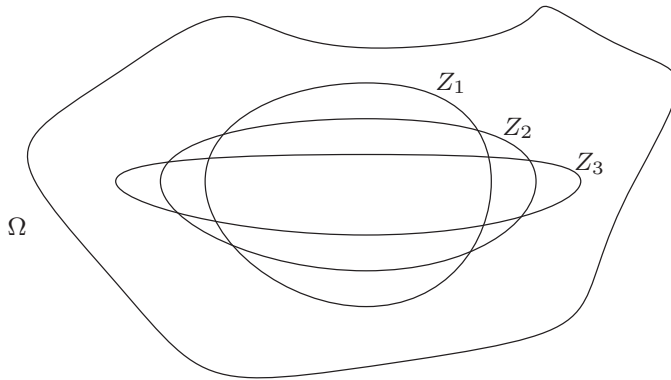
FIGURE 8.4. An example where three frequencies are needed in two dimensions. For simplicity, only the constraint $|u_\omega^1(x)| \geq C$ is considered. The same notation of Figure 8.3 is used.

want to satisfy the constraints everywhere. Indeed, the assumptions given here are motivated by the relevant elliptic regularity estimates, which are known to be optimal. In particular, if a is not Hölder continuous, the gradient of u_ω^i may not be well-defined everywhere.

▷ The main advantage of this approach is in the explicit construction of simple boundary conditions, independently of a (if $d = 2$) and of ε and σ.

▷ The main disadvantage is the need of multiple frequencies, and so this approach can be applied only with frequency-dependent PDE. Even though they are determined *a priori* and independently of the parameters, many more measurements may be needed when compared to other methods.

▷ As mentioned in the previous section, the proof of these theorems is based on a reduction to the zero-frequency case, where the constraints can be easily enforced. This feature is shared also with the approaches based on CGO and the Runge approximation, which are based on a reduction to the Laplace equation or to a constant-coefficient PDE, respectively.

8.3. Proofs of the main results

We start with the well-posedness and regularity of (8.4); the result is classical.

Lemma 8.5. — *Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,\alpha}$ bounded domain for some $\alpha \in (0,1)$ and $d \in \{2,3\}$. Assume that $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy (8.5), (8.6) and (8.7) for some $\Lambda > 0$. There exist $\eta, C > 0$ depending only on Ω, α, Λ and $K_{\max}$ such that the following is true. Set*

$$E_\eta = \big\{ z \in \mathbb{C} : |\operatorname{Re} z| < K_{\max} + 1, |\operatorname{Im} z| < \eta \big\}.$$

For every $\omega \in E_\eta$, $\varphi \in C^{1,\alpha}(\overline{\Omega}; \mathbb{C})$, $F \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^d)$ and $f \in L^\infty(\Omega; \mathbb{C})$ the problem

$$\begin{cases} -\operatorname{div}(a\nabla u) - (\omega^2\varepsilon + i\omega\sigma)u = f + \operatorname{div} F & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution $u \in H^1(\Omega; \mathbb{C})$. Moreover, $u \in C^{1,\alpha}(\overline{\Omega}; \mathbb{C})$ and

$$\|u\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})} \le C\Big(\|f\|_{L^\infty(\Omega;\mathbb{C})} + \|F\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^d)} + \|\varphi\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})} \Big).$$

For $\varphi \in C^{1,\alpha}(\overline{\Omega}; \mathbb{C})$ and $\omega \in E_\eta$ let u_ω^φ be the unique solution to

$$\begin{cases} -\operatorname{div}(a\nabla u_\omega^\varphi) - (\omega^2\varepsilon + i\omega\sigma)u_\omega^\varphi = 0 & \text{in } \Omega, \\ u_\omega^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

Note that by the previous result we have

(8.9) $$\|u_\omega^\varphi\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})} \le C\|\varphi\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})}$$

for some $C > 0$ depending only on Ω, α, Λ and $K_{\max}$.

As we have already pointed out in Section 8.1, at the core of this approach is the holomorphicity of the map $\omega \in E_\eta \mapsto u_\omega^\varphi \in C^1(\overline{\Omega}; \mathbb{C})$.

Proposition 8.6. — *Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,\alpha}$ bounded domain for some $\alpha \in (0,1)$ and $d \in \{2,3\}$. Assume that $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{d \times d})$ and $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$ satisfy (8.5), (8.6) and (8.7) for some $\Lambda > 0$. Take $\varphi \in C^{1,\alpha}(\overline{\Omega}; \mathbb{C})$, $K_{\max} > 0$ and let $\eta > 0$ be as in Lemma 8.5. The map*

$$E_\eta \longrightarrow C^1(\overline{\Omega}; \mathbb{C}), \quad \omega \longmapsto u_\omega^\varphi$$

is holomorphic. Moreover $\partial_\omega u_\omega^\varphi \in C^1(\overline{\Omega}; \mathbb{C})$ is the unique solution to

$$(8.10) \quad \begin{cases} -\operatorname{div}(a\nabla(\partial_\omega u_\omega^\varphi)) - (\omega^2 \varepsilon + \mathrm{i}\omega\sigma)\,\partial_\omega u_\omega^\varphi = (2\omega\varepsilon + \mathrm{i}\sigma)u_\omega^\varphi & \text{in } \Omega, \\ \partial_\omega u_\omega^\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. — Fix $\omega_0 \in E_\eta$: we shall prove that $\omega \in E_\eta \mapsto u_\omega^\varphi \in C^1(\overline{\Omega}; \mathbb{C})$ is holomorphic in ω_0. Let $R > 0$ be such that the complex ball $B(\omega_0, R) \subseteq E_\eta$ and take $h \in B(0, R) \subseteq \mathbb{C}$. By Lemma 8.5, the above problem is well-posed with $\omega = \omega_0 + h$. By construction we have

$$-\operatorname{div}(a\nabla u_{\omega_0+h}^\varphi) - \big((\omega_0+h)^2\varepsilon + \mathrm{i}\,(\omega_0+h)\sigma\big)u_{\omega_0+h}^\varphi = -\operatorname{div}(a\nabla u_{\omega_0}^\varphi) - (\omega_0^2\varepsilon + \mathrm{i}\,\omega_0\sigma)u_{\omega_0}^\varphi,$$

and $u_{\omega_0+h}^\varphi = u_{\omega_0}^\varphi$ on $\partial\Omega$. Setting $v_h = (u_{\omega_0+h}^\varphi - u_{\omega_0}^\varphi)/h$ we obtain

$$\begin{cases} -\operatorname{div}(a\nabla v_h) - (\omega_0^2\varepsilon + \mathrm{i}\,\omega_0\sigma)v_h = (2\omega_0\varepsilon + \mathrm{i}\,\sigma)u_{\omega_0+h}^\varphi & \text{in } \Omega, \\ v_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 8.5 and (8.9) give

$$(8.11) \qquad \|v_h\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})} \le C\big\|(2\omega_0\varepsilon + \mathrm{i}\sigma)u_{\omega_0+h}^\varphi\big\|_{L^\infty(\Omega;\mathbb{C})} \le C_1\|\varphi\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})}$$

for some $C_1 > 0$ depending only on Ω, α, Λ and $K_{\max}$.

Defining $\partial_\omega u_{\omega_0}^\varphi$ as in (8.10) and setting $r_h = v_h - \partial_\omega u_{\omega_0}^\varphi$ we obtain

$$\begin{cases} -\operatorname{div}(a\nabla r_h) - (\omega_0^2\varepsilon + \mathrm{i}\,\omega_0\sigma)r_h = (2\omega_0\varepsilon + \mathrm{i}\,\sigma)h v_h & \text{in } \Omega, \\ r_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as above, Lemma 8.5 and (8.11) yield

$$\|r_h\|_{C^1(\overline{\Omega};\mathbb{C})} \le C|h| \cdot \|(2\omega_0\varepsilon + \mathrm{i}\sigma)v_h \cdot \|_{L^\infty(\Omega;\mathbb{C})} \le C_2\|\varphi\|_{C^{1,\alpha}(\overline{\Omega};\mathbb{C})} \cdot |h|$$

for some $C_2 > 0$ depending only on Ω, α, Λ and $K_{\max}$. In other words,

$$\lim_{h\to 0} \frac{u_{\omega_0+h}^\varphi - u_{\omega_0}^\varphi}{h} = \partial_\omega u_{\omega_0}^\varphi \quad \text{in } C^1(\overline{\Omega};\mathbb{C}).$$

This shows that the map

$$E_\eta \longrightarrow C^1(\overline{\Omega};\mathbb{C}), \quad \omega \longmapsto u_\omega^\varphi$$

is holomorphic in ω_0, and that the first derivative with respect to ω solves (8.10), as desired. $\qquad\square$

Choose now the $d+1$ boundary values
$$\varphi_1 = 1, \quad \varphi_2 = x_1, \quad \ldots \quad , \varphi_{d+1} = x_d,$$
(as in Theorems 8.1 and 8.2). In order to study the constraints considered in this chapter, we use the following notation.

For $j = 1, 2, 3$ define the maps $\theta^j \colon E_\eta \to C^0(\overline{\Omega}; \mathbb{C})$ by
$$\theta^1_\omega = u^1_\omega, \quad \theta^2_\omega = \det[\nabla u^2_\omega \cdots \nabla u^{d+1}_\omega], \quad \theta^3_\omega = \det \begin{bmatrix} u^1_\omega & \cdots & u^{d+1}_\omega \\ \nabla u^1_\omega & \cdots & \nabla u^{d+1}_\omega \end{bmatrix},$$
where $\eta > 0$ is given by Lemma 8.5. As an immediate consequence of the previous result we obtain the following

Lemma 8.7. — *There exists $C > 0$ depending only on Ω, α, Λ and $K_{\max}$ such that for every $j = 1, 2, 3$ and $\omega \in E_\eta$*

1) *the map $\theta^j \colon E_\eta \to C^0(\overline{\Omega}; \mathbb{C})$ is holomorphic;*
2) $\|\theta^j_\omega\|_{C^0(\overline{\Omega};\mathbb{C})} \leq C$ *and*
3) $\|\partial_\omega \theta^j_\omega\|_{C^0(\overline{\Omega};\mathbb{C})} \leq C$.

Proof. — Part 1 follows immediately from the holomorphicity of the maps
$$E_\eta \longrightarrow C^1(\overline{\Omega}; \mathbb{C}) \quad \omega \longmapsto u^i_\omega$$
proven in Proposition 8.6, since composition of holomorphic functions is holomorphic. Part 2 follows from (8.9), and Part 3 follows from (8.9) and the estimate

(8.12) $$\|\partial_\omega u^i_\omega\|_{C^1(\overline{\Omega};\mathbb{C})} \leq C(\Omega, \alpha, \Lambda, K_{\max}),$$

which is a consequence of (8.9) and of Lemma 8.5 applied to (8.10). $\square$

We first prove that the constraints can be satisfied in every point x of the domain for some frequency $\omega_x \in \mathcal{A}$. For simplicity, we carry out the proofs of the two theorems at the same time. Thus, in the case of Theorem 8.2, we let $\Omega' = \Omega$.

Proposition 8.8. — *Assume that the assumptions of either Theorem 8.1 or of Theorem 8.2 hold true. For every $x \in \overline{\Omega}'$ and $j = 1, 2, 3$ there exists $\omega \in \mathcal{A}$ such that*
$$\left| \theta^j_\omega(x) \right| \geq C$$
for some $C > 0$ depending only on Ω, Ω', α, Λ and $\mathcal{A}$.

Proof. — Let us first study the values of the maps θ^j for $\omega = 0$. Since $\varphi_1 = 1$, we have $u_0^1 \equiv 1$ independently of the dimension. Therefore

$$\theta_0^1 \equiv 1 \quad \text{and} \quad \theta_0^3 \equiv \theta_0^2.$$

If a is not constant and $d = 2$, by Corollary 6.8 we have

$$|\theta_0^2(x)| \geq C_0, \quad x \in \overline{\Omega'}$$

for some $C_0 \in (0,1]$ depending only on Ω, Ω', α and Λ. If a is constant, since $u_0^i \equiv x_{i-1}$ for $i = 2, \ldots, d+1$ we have $\theta_0^2 \equiv 1$. To summarise the above discussion, we have proved that

$$\left|\theta_0^j(x)\right| \geq C_0, \quad j = 1, 2, 3, \ x \in \overline{\Omega'}.$$

For $x \in \overline{\Omega'}$ define $g_x \colon E_\eta \to \mathbb{C}$ by $g_x(\omega) = \theta_\omega^1(x)\theta_\omega^2(x)\theta_\omega^2(x)$. We have

$$\left|g_x(0)\right| \geq C_0^3, \quad x \in \overline{\Omega'},$$

and by Lemma 8.7 part 2 we have

$$\sup_{E_\eta} |g_x| \leq D$$

for some $D > 0$ depending only on Ω, α, Λ and $\mathcal{A}$. Moreover, in view of Lemma 8.7 part 1, g_x is holomorphic. Thus, by Proposition 8.9 there exists $\omega \in \mathcal{A}$ such that

$$\left|\theta_\omega^1(x)\theta_\omega^2(x)\theta_\omega^2(x)\right| = \left|g_x(\omega)\right| \geq C$$

for some $C > 0$ depending only on Ω, Ω', α, Λ and $\mathcal{A}$. The result immediately follows from Lemma 8.7 part 2. □

We have proven that the required constraints can be satisfied in every point of the domain. This would still require an infinite number of frequencies to enforce the constraints everywhere in the domain. A relatively standard compactness argument allows us to show that in fact a finite number of frequency is sufficient, concluding the argument.

Proof of Theorems 8.1 and 8.2. — Several positive constants depending only on Ω, Ω', α, Λ and $\mathcal{A}$ will be denoted by $C_1, C_2, \ldots$.

In view of Proposition 8.8, for every $x \in \overline{\Omega'}$ and $j = 1, 2, 3$ there exists $\omega_x \in \mathcal{A}$ such that
$$\left| \theta_{\omega_x}^j(x) \right| \geq C_1.$$
By Lemma 8.7 part 3, we have that the partial derivative with respect to ω of θ_ω^j is bounded by above, namely $|\partial_\omega \theta_\omega^j| \leq C_2$. Therefore, the above inequality with a different constant C_3 holds also in a neighbourhood of ω_x, whose size is independent of x. More precisely, there exists $Z > 0$ depending only on Ω, Ω', α, Λ and $\mathcal{A}$ such that for every $x \in \overline{\Omega'}$ and $j = 1, 2, 3$

$$(8.13) \qquad \left| \theta_\omega^j(x) \right| \geq C_3, \quad \omega \in [\omega_x - Z, \omega_x + Z] \cap \mathcal{A}.$$

Write
$$\mathcal{A} = \bigcup_{p=1}^{P} I_p \cap \mathcal{A}, \quad I_p = \left[K_{\min} + (p-1)Z, K_{\min} + pZ \right],$$
for some $P \in \mathbb{N}$ depending only on Z and $\mathcal{A}$. Recall that the set of frequencies $K^{(n)}$ is defined by
$$K^{(n)} = \left\{ K_{\min} + \frac{\ell - 1}{n - 1}(K_{\max} - K_{\min}) : \ell = 1, \ldots, n \right\}.$$
Since the distance between two consecutive frequencies in $K^{(n)}$ goes to zero as $n \to \infty$ and the size of I_p is equal to Z, it is possible to choose n big enough (depending on Z and $\mathcal{A}$) so that $K^{(n)}$ intersects I_p for every $p = 1, \ldots, P$. We can thus write $\omega(p) \in K^{(n)} \cap I_p$.

Fix now $x \in \overline{\Omega'}$. Since the set $[\omega_x - Z, \omega_x + Z]$ has size $2Z$ and the sets I_p, of size Z, cover $\mathcal{A}$, there exists $p_x = 1, \ldots, P$ such that $I_{p_x} \subseteq [\omega_x - Z, \omega_x + Z]$. Therefore $\omega(p_x) \in [\omega_x - Z, \omega_x + Z] \cap \mathcal{A}$, and thanks by (8.13) we obtain

$$(8.14) \qquad \left| \theta_{\omega_{p_x}}^j(x) \right| \geq C_3, \quad j = 1, 2, 3.$$

Define now for $\omega \in K^{(n)}$
$$\Omega_\omega = \left\{ x \in \overline{\Omega'} : |\theta_\omega^j(x)| > \tfrac{1}{2} C_3, \ j = 1, 2, 3 \right\}.$$

The desired constraints are satisfied in Ω_ω by definition of θ^j with the constant $\tfrac{1}{2}C_3$. Moreover, by (8.14) we have

$$\overline{\Omega'} = \bigcup_{\omega \in K^{(n)}} \Omega_\omega,$$

as desired. This concludes the proof of the theorems. $\qquad \square$

8.4. Quantitative unique continuation for holomorphic functions

We need the following quantitative version of the unique continuation property.

Proposition 8.9. — *Take $\eta, C_0, D > 0$ and $0 < K_{\min} < K_{\max}$. Set*

$$\mathcal{A} = [K_{\min}, K_{\max}] \quad and \quad E = \{z \in \mathbb{C} : |\operatorname{Re} z| < K_{\max} + 1, |\operatorname{Im} z| < \eta\}.$$

There exists $C > 0$ such that for every holomorphic function $g \colon E \mapsto \mathbb{C}$ with $\sup_E |g| \leq D$ and $|g(0)| \geq C_0$ we have

$$\max_{\mathcal{A}} |g| \geq C.$$

Proof. — By contradiction, assume there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of holomorphic functions on E such that for every n

$$(8.15) \qquad\qquad \sup_E |g_n| \leq D,$$

$$(8.16) \qquad\qquad |g_n(0)| \geq C_0,$$

$$(8.17) \qquad\qquad \lim_{n \to \infty} \max_{\mathcal{A}} |g_n| = 0.$$

By (8.15) and Montel's theorem [**201**, Chapter 8, Theorem 3.3] there exists a subsequence, still denoted by g_n, and a holomorphic function $g \colon E \mapsto \mathbb{C}$ such that $g_n \to g$ uniformly on every compact subsets of E. As a consequence, in view of (8.16) and (8.17) we have $|g(0)| \geq C_0 > 0$ and $g(z) = 0$ for every $z \in \mathcal{A}$, which contradicts the unique continuation theorem for holomorphic functions. $\qquad\square$

PART II

HYBRID INVERSE PROBLEMS

CHAPTER 9

THE COUPLED STEP IN HYBRID INVERSE PROBLEMS

This part of the book focuses on hybrid inverse problems. Typically, the reconstruction in hybrid imaging techniques is split into two steps.

▷ *The coupled step.* — By combining two different types of waves, some internal data are reconstructed inside the domain from the direct measurements (usually taken on the boundary of the domain considered). The physical realisations of this combination are different for each modality; the mathematical techniques employed to reconstruct the internal data from the measurements vary accordingly. The internal data do not provide values for the unknown parameter explicitly; it measures it indirectly by providing the value of a conglomerate expression involving other quantities, such as the solutions of the direct problem.

▷ *The quantitative step.* — The quantitative step is devoted to the reconstruction of the unknown parameter(s) from the measured internal data. Unlike the coupled step, this is a solely mathematical step which often involves the study of the PDE governing the problem to obtain uniqueness, stability and, in some situations, explicit reconstruction formulae.

This chapter is devoted to the presentation of the coupled step of the hybrid inverse problems introduced in Chapter 1. This chapter does not contain theorems or propositions; some of the tools introduced in Part I of this book find their application here in an informal way. The quantitative step will be the focus of Chapter 10 .

9.1. Magnetic resonance electric impedance tomography – current density impedance imaging

9.1.1. Physical model. — The two modalities we consider here refer to the same physical coupling method. A brief description of this problem follows, and readers are referred to [**196, 195**] for a more detailed discussion.

A conductive body is equipped with the standard EIT apparatus discussed in §1.1.1. For every electric potential φ applied on the boundary of a domain Ω, the corresponding potential u inside Ω satisfies the conductivity equation

$$(9.1) \qquad \begin{cases} -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where σ is the spatially varying conductivity. As a result, a current of the form

$$J = \sigma \nabla u \quad \text{in } \Omega$$

is created inside the domain. The presence of the electrical current creates a magnetic field H, which, by Ampère's law, satisfies

$$(9.2) \qquad\qquad J = \operatorname{curl} H \quad \text{in } \Omega.$$

In a more general and realistic model, we can also consider the full Maxwell system

$$(9.3) \qquad \begin{cases} \operatorname{curl} E = i\omega H & \text{in } \Omega, \\ \operatorname{curl} H = -i\gamma E & \text{in } \Omega, \\ E \times \nu = \varphi \times \nu & \text{on } \partial\Omega. \end{cases}$$

with complex unknown admittance $\gamma = \omega\varepsilon + i\sigma$. We assume $\mu = 1$ and $\varepsilon, \sigma > 0$, namely we study the isotropic case. Note that (9.1) and (9.2) are nothing else than (9.3) in the limit $\omega \to 0$ (Remark 3.20).

9.1.2. The internal data. — The coupled step of these hybrid modalities consists in the reconstruction of one or more components of the magnetic field H with a magnetic resonance imaging (MRI) scanner. As this is a very classical medical imaging modality, we shall not discuss the mathematical details. In vague terms, the reconstruction in MRI boils down to an inversion of the Fourier transform.

If only one component is measured, the modality is usually called *magnetic resonance electric impedance tomography* (MREIT). In order to measure the full magnetic field H, two rotations of the object or of the scanner are required. In this case, the modality takes the name of *current density impedance imaging* (CDII), since by (9.2) the full current J can be easily obtained from H. For simplicity, in this book we shall consider only the case of CDII, even though MREIT is arguably much more practical.

In CDII modelled by the conductivity equation, in view of (9.2) the internal data are given by the current density

$$J = \sigma \nabla u,$$

corresponding to one or several boundary potentials φ. These internal measurements represent the data obtained from CDII: the desired unknown σ is multiplied by the field ∇u.

In the more general case of Maxwell's system (9.3), the internal data simply consist of several measurements of H for several boundary values φ, and both ε and σ are unknown.

In Chapter 10 , we will study the quantitative step in CDII, namely how to reconstruct σ (and ε) from these internal data.

9.2. Acousto-electric tomography

9.2.1. Physical model. — Acousto-electric tomography is a hybrid modality using the electro-acoustic interaction phenomenon, experimentally measured in [135]. It has been developed under different names, such as acousto-electric tomography [218], [101] or ultrasound current source density imaging [175] (and possibly other names as well), and in the mathematical literature as electrical impedance tomography by elastic deformation, impedance acoustic-tomography or ultrasound modulated electrical impedance tomography [25], [99], [71], [129], [40] (and possibly other names as well). The fundamental physical mechanism used for this imaging phenomenon is that when a tissue is compressed, its conductivity is affected. Namely, if a volume D is subject to a

variation of pressure δp, its conductivity varies by

(9.4) $$\delta\sigma \approx \sigma\,k\,\delta p,$$

where k is a proportionality constant [119], [135].

　We will describe one of these imaging modalities, which uses focused waves. In that case small domains D are perturbed by means of focused ultrasound waves. Another possibility is the use of modulated plane waves. The resulting internal data have the same form in both cases.

9.2.2. The internal data. — Let Ω be a smooth three-dimensional domain. (This approach works in the two-dimensional case as well, but we decided to restrict ourselves to three dimensions since the theoretical result of Chapter 4 we are going to use was discussed only for $d = 3$.) We consider two measurements. In the first case, no pressure is applied, and the voltage potential u_i is given by

$$\begin{cases} \operatorname{div}(\sigma\,\nabla u_i) = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where φ_i, $i = 1, \ldots N$, are the imposed boundary voltage potentials for some $N \in \mathbb{N}^*$ (a boundary current can be imposed instead, leading to the same final result). Second, when a focused ultrasound beam is applied to the object, calibrated so that it is centred around a point z located within the domain Ω, the voltage potential satisfies

$$\begin{cases} \operatorname{div}(\sigma^z\nabla u_j^z) = 0 & \text{in } \Omega, \\ u_j^z = \varphi_j & \text{on } \partial\Omega, \end{cases}$$

where $\sigma^z = \sigma + (\delta\sigma)\,\mathbb{1}_{D_z} = \sigma(1 + k\,\delta p\,\mathbb{1}_{D_z})$.

　$D_z = z + D$ is the locus of the focused wave, and D is a set of small diameter centred at the origin. An integration by parts shows that the cross-correlation of the known boundary measurements leads to some localised information on the inclusion, namely

$$\int_{\partial\Omega} \sigma\,(\varphi_i\,\partial_\nu u_j^z - \varphi_j\,\partial_\nu u_i)\,\mathrm{d}s = \int_{D_z} \delta\sigma\,\nabla u_j^z \cdot \nabla u_i\,\mathrm{d}x.$$

We can now apply the theory developed in Chapter 4. More precisely, assuming that $\sigma \in W^{1,p}$ for some $p > 3$, by Theorem 4.7 we have

$$\int_{\partial\Omega} \sigma(\varphi_i \partial_\nu u_j^z - \varphi_j \partial_\nu u_i)\,\mathrm{d}s = \int_{D_z} \delta\sigma(I_3 + P_D^\sigma)\nabla u_j \cdot \nabla u_i\,\mathrm{d}x + O\big(|D|^{1+\delta}\big)$$

for some $\delta > 0$. In particular, if D is a small ball centred at the origin, we derive

$$\frac{1}{|D|}\int_{\partial\Omega}\sigma(\varphi_i\partial_\nu u_j^z - \varphi_j\partial_\nu u_i)\,\mathrm{d}s \approx \frac{\delta\sigma(z)3\sigma(z)}{3\sigma(z) + \delta\sigma(z)}\nabla u_j(z)\cdot\nabla u_i(z)$$
$$= \frac{3k\delta p}{3 + k\delta p}\sigma(z)\nabla u_j(z)\cdot\nabla u_i(z).$$

Varying the applied pressure or otherwise, the constant k can be reconstructed and we obtain

$$\sigma(z)\nabla u_i(z)\cdot\nabla u_j(z)$$

for every centre point z. In other words, the cross-correlation of the boundary measurements allows us to measure pointwise (cross-)power measurements for the unperturbed problem. If we assume that it is possible to perform the above measurements for all $z \in \Omega'$, for some subdomain $\Omega' \subseteq \Omega$, the internal data in AET is

$$H_{ij}(x) = \sigma(x)\nabla u_i(x)\cdot\nabla u_j(x), \quad x \in \Omega'.$$

Quantitative AET, that is, the reconstruction of σ from multiple measurements of H_{ij} will be addressed in the next chapter.

9.3. Thermoacoustic tomography

9.3.1. Physical model. — Thermoacoustic tomography (TAT) is one of the most commonly studied hybrid imaging problem in the mathematical literature of the last decade. Electromagnetic radiations are coupled with ultrasound measurements as we now describe. The absorption of the electromagnetic waves inside the object under investigation results in local heating, and so in a local expansion of the medium. This creates acoustic waves that propagate up to the boundary of the domain, where they can be measured. The frequency of the waves is typically in the microwave range; when high frequency waves, namely laser pulses, are used, this hybrid modality is called photoacoustic tomography (see Section 9.6).

The model assumes that the object under consideration has the mechanical properties of a mostly inviscid fluid. In the case of soft biological tissues, this assumption is reasonable, even though more advanced models prefer to consider visco-elastic tissues instead.

Assuming that the velocities, variations of pressure, and variations of densities are sufficiently small to justify a linearised model, we write down the conservation of mass and momentum under the form

$$\frac{1}{\rho}\frac{d\rho}{dt} + \operatorname{div} U = 0, \quad \rho\frac{d}{dt}u = \sum_{j=1}^{d}\partial_j\sigma_{ij},$$

in absence of external forces [89, Chapter 1]. The velocity vector is u in Eulerian coordinates, and U is the same vector in Lagrangian coordinates, ρ is the density, and σ_{ij} is the stress tensor. The constitutive equation defining the stress is

$$\sigma_{ij} = \big(-p + \lambda \operatorname{div}(u)\big)\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j) + \rho F_{ij}(x,\theta),$$

where p is the pressure, λ and μ are the coefficients of viscosity, and F is the thermal stress tensor, accounting for the effects of the temperature θ. It is usually assumed that viscosity can be neglected and that the thermal stress tensor isotropic, leading to a simpler (Euler) model,

$$\rho\frac{du_i}{dt} = -\partial_i\big(p + \rho F(x,\theta)\big).$$

Assuming the flow is irrotational, that is, $\operatorname{curl} U = 0$, the velocities derive from a potential,

$$U = \nabla\varphi,$$

and the problem becomes

$$\partial_t p + \nabla\rho \cdot U + \rho\Delta\varphi = 0, \quad \partial_i\big(\partial_t\varphi + \tfrac{1}{2}U^2\big) + \frac{1}{\rho}\partial_i\big(p + \rho F(x,\theta)\big) = 0.$$

Linearising again, we obtain

$$\partial_t\rho + \rho\Delta\varphi = 0, \quad \partial_i\left(\partial_t\varphi + \frac{1}{\rho}p + F(x,\theta)\right) = 0,$$

e.g.

(9.5) $$\partial_t\varphi + \frac{1}{\rho}p + F(x,\theta) = 0,$$

absorbing the constant (independent of time) into φ. Assuming that the density depends on the location, the pressure and the temperature according to an equation of state, namely

$$\rho = \rho(p, x, \theta).$$

As a result, the first equation becomes

$$\frac{\partial \rho}{\partial p} \partial_t p + \frac{\partial \rho}{\partial \theta} \partial_t \theta + \rho \Delta \varphi = 0,$$

Combining these two equations gives after linearisation

$$\frac{\partial \rho}{\partial p} \partial_{tt} \varphi - \Delta \varphi = \partial_t \theta \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} - \frac{\partial \rho}{\partial p} \partial_\theta F(x, \theta) \right).$$

The temperature θ is assumed to satisfy the heat equation,

$$\partial_t \theta - \frac{1}{\rho} \operatorname{div}(K \nabla \theta) = S(x, t),$$

where $S(x, t)$ is a source term accounting for the heat added to the system by electromagnetic radiations. The commonly accepted model is a particular case of this system, where the thermal diffusion is considered to happen at a larger time scale than the propagation of the the pressure wave, therefore

$$\partial_t \theta = S(x, t),$$

and the problem finally becomes

$$\frac{\partial \rho}{\partial p} \partial_{tt} \varphi - \Delta \varphi = S(x, t) \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} - \frac{\partial \rho}{\partial p} \partial_\theta F(x, \theta) \right).$$

It is usually assumed that F is linear in θ, that is $\partial_\theta F(x, \theta) = \beta(x)$. Using (9.5) we obtain

$$\frac{\partial \rho}{\partial p} \frac{1}{\rho} \partial_{tt} p - \Delta \left(\frac{1}{\rho} p \right) = \partial_t S(x, t) \left(\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} - 2 \frac{\partial \rho}{\partial p} \beta \right) + \Delta F.$$

It is further assumed that the spatial variations of ρ and F are so mild that this model may be simplified to

$$c^{-2} \partial_{tt} p - \Delta p = \partial_t S(x, t) A(x),$$

for some function A. From thermodynamic considerations, the source term $S(x,t)$ is seen to be proportional to the Joule energy deposited by the electromagnetic radiating field, *e.g.* (see [208])

$$S(x,t) = \frac{1}{C_p}\sigma(x)\big|E(x,t)\big|^2.$$

If the illumination happens suddenly, so that S is modelled as an initial impulse, this leads to the commonly accepted model, namely

$$\begin{cases} \partial_{tt}p - c^2(x)\Delta p = 0 & \text{in } \Omega \times (0,T), \\ p(x,0) = \Gamma\sigma(x,\omega_c)|E(x,\omega_c)|^2 & \text{in } \Omega, \\ \partial_t p(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where Γ is a mildly varying function of space, the so-called *Grüneisen parameter*. The boundary conditions satisfied by the pressure depend on the approach followed to reconstruct the source from boundary measurements. One possibility is to consider idealised acoustic receptors, "invisible" to the acoustic propagation, situated at a certain distance from the medium. In this case, this equation holds in the whole space. In other words, $\Omega = \mathbb{R}^d$ and no further boundary conditions are imposed. Then, one assumes that the initial source has support contained in some bounded domain $\Omega' \subseteq \mathbb{R}^d$, and that the acoustic measurements are performed on $\partial\Omega'$. (For the partial data problem, only a subset of $\partial\Omega'$ is considered.) This is the most studied setting in the mathematical literature: a good understanding of this problem has now been reached, and a successful inversion is often possible, even with partial data or non-constant sound speed. The reader is referred to [127] for a review on the main advances related to this inverse problem.

In this book we follow a different approach: the wave propagation is considered only within the bounded domain surrounded by the sensor surface, and assume a certain behaviour of the acoustic wave at the boundary (see [84], [26], [130], [116], [1], [200]). In other words, we set $\Omega = \Omega'$ and we augment the previous initial boundary problem with suitable boundary conditions on $\partial\Omega$. For simplicity, here we choose Dirichlet boundary conditions:

$$p(x,t) = 0 \quad \text{in } \partial\Omega \times [0,T].$$

Other types of boundary conditions may be considered as well; for instance, Neumann boundary conditions would be appropriate for a reflecting cavity. In the Dirichlet case, the measurements are

$$\partial_\nu p(x,t), \quad x \in \Sigma, \ t \in [0,T]$$

for some measuring surface $\Sigma \subseteq \partial\Omega$ and some time $T > 0$.

9.3.2. The internal data. — From the previous discussion, the acoustic pressure satisfies the wave equation

$$\begin{cases} c(x)^2 \Delta p - \partial_{tt}^2 p = 0 & \text{in } \Omega \times (0,T), \\ p(x,0) = H(x) & \text{in } \Omega, \\ \partial_t p(x,0) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$, $d = 2,3$, is a smooth bounded domain, c is the sound speed of the medium and H is the initial source. By the theory developed in Chapter 2, see Remark 2.7, it is possible to reconstruct the source term H from boundary measurements of $\partial_\nu p$ on a part of the boundary $\Sigma \subseteq \partial\Omega$, provided that the observability inequality is satisfied. In the sequel, we assume that the observability inequality holds.

Assuming $\Gamma = 1$, the source term H has the form

$$H(x) = \sigma(x)\big|E(x)\big|^2,$$

where σ is the spatially varying conductivity of the medium and E is the electric field that satisfies the Maxwell system

$$-\operatorname{curl}\operatorname{curl} E + (\omega^2 + i\omega\sigma)E = 0,$$

where $\omega > 0$ is the angular frequency of the microwaves.

We consider only a simplified version of the above model, namely the standard scalar approximation of Maxwell's system given by the Helmholtz equation

$$\begin{cases} \Delta u + (\omega^2 + i\omega\sigma)u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

For simplicity, we have augmented the problem with Dirichlet boundary values, even though Robin boundary conditions would be arguably more appropriate

in this context. In this simplified scalar case, the measured absorbed energy takes the form

$$H(x) = \sigma(x)\big|u(x)\big|^2, \quad x \in \Omega.$$

This quantity represents the so-called thermoacoustic image.

In the following chapter (see Section 10.3) we shall deal with the problem of quantitative thermoacoustics, namely the problem of reconstructing σ from the knowledge of the internal data H. There exists an explicit reconstruction formula for the reconstruction of σ, provided that several measurements are taken for different boundary values φ.

9.4. Dynamic elastography

In elastography, the medium is modelled as a solid instead of a fluid. The variable u now represents the displacement and not the velocity. The classical isotropic dynamic linear elasticity model, resulting from Newton's second law, is

$$\rho\frac{\partial^2 u}{\partial t^2} = f + \operatorname{div}\sigma(u)$$

where $\sigma = (\sigma_{ij})_{1\leq i,j\leq 3}$ is the stress tensor, related to the displacement u by Hooke's Law

$$\sigma(u) = \lambda\operatorname{div}(u)I_d + \mu(\nabla u + \nabla u^T),$$

where λ and μ are the Lamé parameters, and I_d is the identity matrix.

The gravity force f is usually neglected compared to other forces in play. This yields

$$\rho\frac{\partial^2 u}{\partial t^2} = \nabla\big(\lambda\operatorname{div}(u)\big) + \operatorname{div}(2\mu\nabla u) + \operatorname{curl}(\mu\operatorname{curl} u),$$

where the operator $\operatorname{div}(2\mu\nabla\cdot)$ acts component-wise, like the vector Laplacian. A *Helmholtz decomposition* (see Lemma 3.7) allows us to decompose u into a compression wave and a shear wave. Namely, we write

$$u = \nabla q + \operatorname{curl}\Phi.$$

The quantity $u_c = \nabla q$ is called the compression wave, whereas $u_s = \text{curl}\,\Phi$ is a called a shear wave, and

$$(9.6) \qquad \rho \frac{\partial^2 u}{\partial t^2} = \nabla\big(\lambda \,\text{div}(u_c)\big) + \text{div}\,\big(2\mu\nabla(u_c + u_s)\big) + \text{curl}(\mu\,\text{curl}\,u_s).$$

When μ and λ are constant, this becomes simply

$$(9.7) \qquad \text{div}\,\sigma(u) = \nabla\big((\lambda + 2\mu)\,\text{div}(u_c)\big) - \text{curl}(\mu\,\text{curl}\,u_s),$$

which acts independently on gradient fields and gradient free fields: this has lead to the separate investigation of compression waves and shear waves. In inhomogeneous media it is often assumed that shear waves and compression waves do not interact at first order, and that (9.7) still holds [4]. Due to the fact that $\lambda \gg \mu$ in tissues, the compression wave and the shear wave are then easily separated after a Fourier transform, as they are deemed to propagate at very different velocities. The fast compression wave is given by

$$\rho\omega^2 u_c + \text{div}\,\big((\lambda + 2\mu)\,\text{div}(u_c)I_d\big) = 0,$$

whereas the slower shear waves satisfies

$$\rho\omega^2 u_s - \text{curl}(\mu\,\text{curl}\,u_s) = 0.$$

The displacement of the shear wave are then measured by either ultra-fast ultrasound imaging, or magnetic resonance [177], [178], [190], [102]. As a result, the available data is u_s in the medium, and the main unknowns of the problem are the functions ρ and μ. In the following chapter we consider the problem of quantitative elastography, which consists of the reconstruction of ρ and μ from several measurements of the internal displacement u_s.

9.5. The thermoelastic problem

We mention here an early model described in [174], which shares similarities with thermoacoustics and elastography. As far as the authors know, it has not evolved into an experimental imaging method yet. We follow the description given in [89, Chapter 1] of the corresponding physical principles.

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4. As far as the authors are aware, this assumption is made to simplify the models; a rigorous justification might be difficult to derive.

Setting u as the small displacement in the medium, and θ a small variation in temperature, assuming Hooke's law of isotropic elasticity holds, with Lamé Parameters $\lambda(x)$ and $\mu(x)$, and setting $\rho(x)$ as the density of the medium, we have

$$\rho(x)\,\partial_{tt}\,u_i = \partial_j(\sigma_{ij}) + f,$$

where f are the external forces (neglected in the sequel), the strain tensor σ is given by

$$\sigma_{ij} := (\lambda(x)\,\mathrm{div}(u) + \alpha\big(2\mu(x) + 3\lambda(x)\big)\theta)\delta_{ij} + 2\mu(x)\mathcal{E}_{ij}(u),$$

where

$$\mathcal{E}(u) = \tfrac{1}{2}\,\partial_j u_i + \tfrac{1}{2}\,\partial_i u_j$$

is the tensor of linearised deformation, α is the linear coefficient of thermal dilatation, and u is the displacement (unlike in the fluid model where it represented a velocity). Assuming as before that the displacement field is irrotational (as it is caused by a dilatation), so that $u = \nabla\varphi$, and because in aqueous tissues $\lambda/\mu \approx 5.10^2$ so that she shear parameter can be neglected, we obtain

$$\sigma_{ij} = (\lambda\Delta\varphi + \alpha\,3\lambda(x)\theta)\delta_{ij},$$

and in turn,

$$\rho(x)\,\partial_{tt}\,\varphi = \lambda\Delta\varphi + \alpha\,3\lambda(x)\theta + G(x),$$

and the integrating factor $G(x)$ can be integrated into φ by setting

$$\Delta g(x) = \frac{G(x)}{\lambda(x)}$$

and redefining as $\widetilde{\varphi} = \varphi + g$. We have obtained

$$\frac{\rho}{\lambda}\,\partial_{tt}\,\varphi - \Delta\varphi = 3\alpha\theta.$$

The heat equation writes

$$\beta\,\partial_t\theta - \mathrm{div}(K\nabla\theta) = 3\lambda\alpha\,\mathrm{div}(\partial_t u) + F(x,t) = 3\lambda\alpha\,\partial_t(\Delta\varphi) + F(x,t),$$

where $\beta = (\rho/T)c_e$, T is the reference temperature, c_e is the specific heat at constant strain, and r represents the rate of heat supplied externally (by radiation in our case), and K is the heat conduction coefficient, divided by T.

Introducing the velocity $\psi = \partial_t \varphi$, linearising, and once again neglecting thermal diffusion as its time-scale is of a different order of magnitude than the speed of wave propagation, we arrive at

$$\frac{\rho}{\lambda} \partial_{tt} \psi - \left(1 + \frac{9\lambda\alpha^2}{\beta}\right)\Delta\psi = 3\alpha\,\partial_t F,$$

that is, a model very similar to the thermoacoustic one (but with a different definition of the c), when linear approximations and irrotational, shear-less assumptions are valid, and the (double) inverse problem is similar.

9.6. Photoacoustic tomography

Photoacoustic tomography (PAT) (sometimes referred to as optoacoustic tomography) and TAT exploit the same physical phenomenon: the propagation of acoustic waves due to the expansions of tissues caused by the absorption of electromagnetic radiation [208]. The only difference lies in the frequency of the EM waves: microwaves for TAT and light (laser) for PAT. As a result, the physical model for PAT is the same as the one described above for TAT, except for the different initial source for the pressure wave. In PAT, this takes the form

$$(9.8) \qquad H(x) = \Gamma(x)\mu(x)u(x), \quad x \in \Omega,$$

where Γ is the Grüneisen parameter, μ is the light absorption and u is the light intensity.

As discussed above for TAT, the acoustic pressure satisfies the wave equation

$$\begin{cases} c(x)^2 \Delta p - \partial_{tt}^2 p = 0 & \text{in } \Omega \times (0, T), \\ p(x, 0) = H(x) & \text{in } \Omega, \\ \partial_t p(x, 0) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega \times (0, T), \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, is a smooth bounded domain, c is the sound speed of the medium and H is the initial source given by (9.8). Arguing as in Section 9.3, in a first step it is possible to reconstruct H from boundary measurements of $\partial_\nu p$ on a part of the boundary $\Sigma \subseteq \partial\Omega$, provided that the observability inequality is satisfied.

After the acoustic inverse problem comes to the optical inverse problem. A model for light propagation is required to tackle this problem. Light propagation could be modelled using the Maxwell system of equations [93]. Simpler (numerically and theoretically) approximations are often preferred, such as the radiative transport equation, that models the propagation of photons, or its diffusion approximation, valid in highly scattering media [35]. For simplicity, in this work we consider only the diffusive regime, namely

$$- \operatorname{div}(D\nabla u) + \mu u = 0,$$

where the diffusion coefficient D depends on the scattering parameter and on the light absorption μ. In general, D is an unknown of the problem.

In the quantitative step of PAT, the light absorption μ has to be recovered from several measurements of $H_i = \Gamma \mu u_i$, corresponding to different light fields u_i. These are obtained with different illuminations φ_i, which play the role of boundary values for the above second order elliptic PDE. This will be discussed in the next chapter.

CHAPTER 10

THE QUANTITATIVE STEP IN HYBRID INVERSE PROBLEMS

This chapter focuses on the quantitative step of the reconstruction for the hybrid imaging modalities introduced in Chapter 1 and presented in Chapter 9. This reconstruction can be usually achieved, at least formally, in absence of noise, by means of exact reconstruction formulae. The applicability of these formulae is guaranteed if the solutions of the direct problem satisfy certain non-zero constraints inside the domain. Such constraints can be enforced by using the methods discussed in Chapters 6, 7 and 8.

A careful analysis of the reconstruction procedure leads to stability estimates in most cases. A detailed discussion of this issue, which is of foremost importance, goes beyond the scope of this book. Note that the reconstruction procedures discussed below typically involve the differentiation of the internal data. In practice, and particularly in presence of noise, these steps must be suitably approximated, *e.g.* via a regularisation procedure. This is a standard issue in inverse problems, and will not be discussed here.

10.1. Current density impedance imaging

This section focuses on the quantitative step of the reconstruction in current density impedance imaging (CDII).

10.1.1. The conductivity equation. — In the first step, we measured the internal current distributions

$$J_i(x) = \sigma(x)\nabla u^i(x), \quad x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ is a Lipschitz bounded domain, $d = 2, 3$, σ is the conductivity of the medium such that $\Lambda^{-1} \leq \sigma \leq \Lambda$ in Ω and the electric potentials u^i satisfy

$$\begin{cases} -\operatorname{div}(\sigma \nabla u^i) = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

For simplicity we shall also assume that σ is known in $\Omega \setminus \Omega'$, for some connected subdomain $\Omega' \Subset \Omega$. The quantitative step of CDII consists of the reconstruction of σ from the knowledge of the internal data J_i. In this book, we shall present a simple direct reconstruction, whose main ideas are taken from [**137**], [**115**].

Let us first discuss the required regularity for the conductivity σ. The reconstruction is based on the differentiation of the data, and so we need $J_i \in H^1(\Omega')$. Moreover, the reconstruction is based on a pointwise non-vanishing condition depending on the first derivatives of u^i, and so we need $u^i \in C^1(\overline{\Omega'}; \mathbb{R})$. The following result gives minimal assumptions on the regularity of σ so that these conditions hold true.

Lemma 10.1. — *If* $\sigma \in H^1(\Omega; \mathbb{R}) \cap C^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ *for some* $\alpha \in (0, 1)$ *then* u^i *belongs to* $C^1(\Omega; \mathbb{R})$ *and* J_i *to* $H^1_{\text{loc}}(\Omega)$.

Proof. — Since $\sigma \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R})$, by classical Schauder estimates (see Lemma 8.5 or [**105**, Corollary 8.36]) we have $u^i \in C^1(\Omega; \mathbb{R})$. Thus, by Lemma 3.2 applied to

$$-\Delta u^i = \sigma^{-1}\nabla\sigma \cdot \nabla u^i \quad \text{in } \Omega,$$

it follows that $u^i \in H^2_{\text{loc}}(\Omega)$. For $j = 1, 2$ we have

$$\partial_j J_i = (\partial_j\sigma)\nabla u^i + \sigma\nabla\partial_j u^i.$$

As a result, $J_i \in H^1_{\text{loc}}(\Omega)$. This concludes the proof. $\qquad\square$

10.1.1.1. *The two dimensional case.* — Assume that the domain Ω is convex. The reconstruction formula is based on two independent measurements such that the matrix $[J_1(x), J_2(x)]$ is invertible in Ω'. Up to a factor $\sigma(x)$, this is equivalent to the non-degeneracy of the Jacobian

$$\det\left[\nabla u^1(x)\ \nabla u^2(x)\right], \quad x \in \Omega'.$$

This brings us to the focus of Chapter 6. In particular, Corollary 6.8 gives that if we choose $\varphi_i = x_i$ for $i = 1, 2$ then

(10.1) $$\left|\det\left[\nabla u^1(x)\ \nabla u^2(x)\right]\right| \geq C, \quad x \in \Omega',$$

for some $C > 0$ depending only on Ω, Ω', Λ and α.

We are now ready to derive a reconstruction formula. Set

$$\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

As $\operatorname{div}(\mathcal{J}\nabla u^i) = 0$ in Ω, we have $\operatorname{div}(\sigma^{-1}\mathcal{J}J_i) = 0$ in Ω. Using the chain rule this becomes $\mathcal{J}J_i \cdot \nabla \log \sigma^{-1} = -\operatorname{div}(\mathcal{J}J_i)$ in Ω. This identity in a more compact form is

$$^T[\mathcal{J}J_1\ \mathcal{J}J_2]\nabla \log \sigma = \operatorname{div}\left(\mathcal{J}[J_1\ J_2]\right).$$

The non-zero constraint (10.1) allows us to invert the matrix $^T[\mathcal{J}j_1\ \mathcal{J}j_2]$, since

$$[\mathcal{J}J_1\ \mathcal{J}J_2](x) = \sigma(x)\mathcal{J}[\nabla u^1(x)\ \nabla u^2(x)], \quad x \in \Omega'.$$

As a consequence, we have the following reconstruction formula.

Proposition 10.2. — *Under the above assumptions, we have*

$$\nabla \log \sigma = {}^T[\mathcal{J}J_1\ \mathcal{J}J_2]^{-1}\operatorname{div}(\mathcal{J}[J_1\ J_2]) \quad \text{in } \Omega'.$$

Since the right-hand side of this identity is known, this equation can be integrated directly along line segments in Ω', thereby obtaining σ in Ω' up to a multiplicative constant, that can be determined if σ is known at one point of Ω'.

If σ is known on the whole $\Omega \setminus \Omega'$, then the reconstruction may be carried out by solving the following Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta v = -\operatorname{div}\left({}^T[\mathcal{J}J_1\ \mathcal{J}J_2]^{-1}\operatorname{div}(\mathcal{J}[J_1\ J_2])\right) & \text{in } \Omega', \\ v = \log \sigma & \text{on } \partial\Omega', \end{cases}$$

and setting $\sigma = e^{v}$ in Ω'. Uniqueness and stability follow immediately from the classical PDE theory (estimate (10.1) gives a stable inversion of the matrix $[\mathcal{J}J_1\ \mathcal{J}J_2]$): well-posedness for this inverse problem is established.

10.1.1.2. *The three dimensional case.* — The reconstruction formula when $d = 3$ is based on a method similar to the one used in the two dimensional case, where the operator div J becomes the standard curl operator in three dimensions.

In order to apply the direct formula we need to have two linearly independent currents at every point. In the two dimensional case, we were able to use the results discussed in Chapter 6. However, such results do not hold in three dimensions, as we have shown in Section 6.5. We therefore use the techniques of Chapter 7 instead. Suitable boundary values will not be determined explicitly, and higher regularity of σ will have to be assumed.

In particular we may use complex geometric optics solutions to construct such illuminations, as discussed in §7.2.3.1. Assume $\sigma \in H^{3/2+3+\delta}(\Omega)$ for some $\delta > 0$. By Theorem 7.2 there exist $\varphi_1, \varphi_2 \in C^2(\overline{\Omega}; \mathbb{R})$ such that ∇u^1 and ∇u^2 are linearly independent in Ω. Hence the corresponding internal data J_1 and J_2 are linearly independent in Ω', namely

$$(10.2) \qquad J_1(x) \times J_2(x) \neq 0, \quad x \in \Omega'.$$

Note that this corresponds to the two-dimensional constraint given in (10.1).

We readily derive

$$0 = \operatorname{curl} \nabla u^i = \operatorname{curl}(\sigma^{-1}J_i) = \sigma^{-1} \operatorname{curl} J_i + \nabla\sigma^{-1} \times J_i \quad \text{in } \Omega,$$

whence $J_i \times \nabla \log \sigma = -\operatorname{curl} J_i$ in Ω. Taking a scalar product with $\mathbf{e}_j$ for $j = 1, 2, 3$ yields

$$\mathbf{e}_j \times J_i \cdot \nabla \log \sigma = -\operatorname{curl} J_i \cdot \mathbf{e}_j \quad \text{in } \Omega.$$

Combining these equations for $i = 1, 2$ and $j = 1, 2, 3$ gives

$$^T[\mathbf{e}_1 \times J_1\ \mathbf{e}_1 \times J_2 \cdots \mathbf{e}_3 \times J_1\ \mathbf{e}_3 \times J_2] \nabla \log \sigma$$
$$= -\,^T[\operatorname{curl} J_1 \cdot \mathbf{e}_1\ \operatorname{curl} J_2 \cdot \mathbf{e}_1 \cdots \operatorname{curl} J_1 \cdot \mathbf{e}_3\ \operatorname{curl} J_2 \cdot \mathbf{e}_3] \quad \text{in } \Omega.$$

We now claim that the matrix $[\mathbf{e}_1 \times J_1 \; \mathbf{e}_1 \times J_2 \cdots \mathbf{e}_3 \times J_1 \; \mathbf{e}_3 \times J_2]$ has rank three in Ω'. Indeed, fix $x \in \Omega'$ and take $z \in \mathbb{R}^3$ such that $\mathbf{e}_j \times J_i(x) \cdot z = 0$ for all i and j. Then $J_i(x) \times z = 0$ for all i, which by (10.2) establishes our claim.

Therefore, we have the following reconstruction formula.

Proposition 10.3. — *Under the above assumptions, we have*

$$\nabla \log \sigma = - {}^T[\mathbf{e}_j \times J_i]_{i,j}^{-1} \cdot {}^T[\operatorname{curl} J_i \cdot \mathbf{e}_j]_{i,j} \quad in \ \Omega'.$$

As in the two dimensional case, this equation can be integrated in Ω', and σ can be reconstructed in Ω' up to a multiplicative constant. Well-posedness for this inverse problem follows as above.

It is worth noting that the suitable boundary conditions may be constructed by means of the Runge approximation, as discussed in Theorem 7.11. The advantage lies in the regularity assumption on σ: since σ is isotropic, it suffices to suppose $\sigma \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}) \cap W^{1,3}(\Omega; \mathbb{R})$ (see Remark 7.6). However, the boundary values constructed with this method provide the invertibility constraints only locally: many more measurements are then required.

10.1.2. Maxwell's equations. — We now consider the model with Maxwell's system of equations

(10.3)
$$\begin{cases} \operatorname{curl} E^i = i\omega H^i & \text{in } \Omega, \\ \operatorname{curl} H^i = -i\gamma E^i & \text{in } \Omega, \\ E^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

The inverse problem we study consists of the reconstruction of the complex valued function $\gamma = \omega\varepsilon + i\sigma$ from the knowledge of internal magnetic fields. In view of the regularity theory for Maxwell's system developed in Chapter 3, we assume that $\varepsilon, \sigma \in C^{0,\alpha}(\Omega; \mathbb{R}) \cap W^{1,3}(\Omega; \mathbb{R})$ for some $\alpha \in (0, \frac{1}{2}]$. By Theorems 3.9 and 3.10, these assumptions guarantee that E^i belongs to $C^{0,\alpha}(\Omega; \mathbb{C}^3) \cap H^1_{\text{loc}}(\Omega; \mathbb{C}^3)$. In particular, we have

$$\operatorname{curl} H^i, E^i \in C^{0,\alpha}(\Omega; \mathbb{C}^3), \quad \operatorname{curl} H^i \in H^1_{\text{loc}}(\Omega; \mathbb{C}^3).$$

We may therefore refer to pointwise values of E^i and differentiate $\operatorname{curl} H^i$ in a weak sense.

We present a straightforward extension of the strategy used previously to address the conductivity equation. We therefore assume that we have two measurements with associated linearly independent electric fields, namely

$$\text{(10.4)} \qquad\qquad E^1(x) \times E^2(x) \neq 0, \quad x \in \Omega'.$$

In order to construct suitable boundary values such that the corresponding solutions satisfy (10.4) (at least locally), different techniques based on complex geometric optics solutions for Maxwell's system or on the use of multiple frequencies may be employed. More precisely, by using the above regularity properties, results similar to those discussed in Chapters 7 and 8 may be derived for the Maxwell system of equations as well. We have decided to omit these developments in this book; the interest reader is referred to [**80**], [**83**] for the CGO approach and to [**6**], [**5**] for the multi-frequency approach.

From the non degeneracy condition (10.4) and Maxwell's system we find

$$\text{(10.5)} \qquad\qquad \operatorname{curl} H^1(x) \times \operatorname{curl} H^2(x) \neq 0, \quad x \in \Omega'.$$

We now proceed to eliminate the unknown electric field from system (10.3), in order to obtain an exhibit an identity involving only ε and σ as unknowns and the magnetic field as a known datum. A computation shows that for $i = 1, 2$,

$$\nabla \gamma \times \operatorname{curl} H^i = \gamma \operatorname{curl} \operatorname{curl} H^i - \gamma^2 \omega H^i \quad \text{in} \ \ \Omega.$$

Projecting these identities along $\mathbf{e}_j$ for $j = 1, 2, 3$ we have

$$\nabla \gamma \cdot \left(\operatorname{curl} H^i \times \mathbf{e}_j \right) = -\gamma (\operatorname{curl} \operatorname{curl} H^i)_j - \gamma^2 \omega (H^i)_j \quad \text{in} \ \ \Omega.$$

We can now write these six equations in a more compact form. By introducing the 3×6 matrix

$$M = \left[\operatorname{curl} H^1 \times \mathbf{e}_1 \ \operatorname{curl} H^2 \times \mathbf{e}_1 \cdots \operatorname{curl} H^1 \times \mathbf{e}_3 \ \operatorname{curl} H^2 \times \mathbf{e}_3 \right]$$

and the six-dimensional horizontal vectors

$$v = \left((H^1)_1, (H^2)_1, \ldots, (H^1)_3, (H^2)_3 \right), \quad w = \left((\operatorname{curl} \operatorname{curl} H^i)_j \right)_{j=1,2,3}^{i=1,2}$$

we obtain

$$\text{(10.6)} \qquad\qquad \nabla \gamma M = -\gamma w - \gamma^2 \omega v \quad \text{in} \ \ \Omega.$$

Arguing as above, in view of (10.5) for $x \in \Omega'$ the matrix $M(x)$ admits a right inverse, which with an abuse of notation we denote by $M^{-1}(x)$. Therefore, problem (10.6) may be rewritten as follows.

Proposition 10.4. — *Under the above assumptions, we have*

$$\nabla \gamma = -\gamma w M^{-1} - \gamma^2 \omega v M^{-1} \quad in \ \Omega'.$$

It is now possible to solve this PDE and reconstruct ε and σ in every $x \in \Omega'$ if these are known for one value $x_0 \in \overline{\Omega'}$.

10.2. Acousto-electric tomography

In this section we study the problem of quantitative AET, namely of reconstructing the conductivity σ from internal data of the form

$$H_{ij}(x) = \sigma(x) \nabla u^i(x) \cdot \nabla u^j(x), \quad x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ is a $C^{1,\alpha}$ bounded domain, $d = 2, 3$, σ is the conductivity of the medium such that $\Lambda^{-1} \leq \sigma \leq \Lambda$ in Ω and the electric potentials u^i satisfy

$$\begin{cases} -\operatorname{div}(\sigma \nabla u^i) = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

for $i = 1, \dots, N$.

This problem has been studied in [25], [71], [129], [40] and solved by using different techniques. These have been extended to the anisotropic case in [160], [162]. In this work, we shall discuss the explicit reconstruction algorithms considered in [71], [40], which are applicable provided that $\{\nabla u^1(x), \dots, \nabla u^N(x)\}$ spans $\mathbb{R}^d$ for every $x \in \Omega'$, and shall follow the presentation of [40]. Therefore, as in the previous section, we will need to apply the techniques discussed in Chapters 6 and 7 for the two-dimensional and three-dimensional cases, respectively.

As with CDII, the reconstruction is based on the differentiation of the data, and so some regularity of H is needed. Assume $\sigma \in W^{1,2}(\Omega; \mathbb{R}) \cap C^{0,\alpha}(\overline{\Omega}; \mathbb{R})$ for some $\alpha \in (0,1)$. By Lemma 10.1 there holds $u^i \in C^1(\Omega; \mathbb{R})$ and $H_{ij} \in H^1_{\text{loc}}(\Omega)$.

10.2.1. Local reconstruction. — This subsection is devoted to the local reconstruction of σ. Let $\widetilde{\Omega} \subseteq \Omega$ be a connected smooth subdomain of Ω and take d boundary values $\varphi_1, \ldots, \varphi_d \in H^1(\Omega; \mathbb{R})$ such that

$$(10.7) \qquad \det \left[\nabla u^1 \cdots \nabla u^d \right] \geq C \quad \text{in } \widetilde{\Omega}.$$

We now prove that σ and ∇u^i can be uniquely reconstructed in $\widetilde{\Omega}$ from the knowledge of H in $\widetilde{\Omega}$ and of $\sigma(x_0)$ and $\nabla u^i(x_0)$ for some $x_0 \in \overline{\widetilde{\Omega}}$.

Write $H_{ij} = S_i \cdot S_j$, where $S_i = \sqrt{\sigma} \nabla u^i$. The reconstruction of σ can be split into two steps:

1) Reconstruction of S_i in $\widetilde{\Omega}$ from the knowledge of H in $\widetilde{\Omega}$ and of $S_i(x_0)$ for some $x_0 \in \widetilde{\Omega}$.
2) Reconstruction of σ in $\widetilde{\Omega}$ from the knowledge of S_i in $\widetilde{\Omega}$ and of $\sigma(x_0)$ for some $x_0 \in \widetilde{\Omega}$.

The second step can be solved by proceeding exactly as in the previous section, where σ was reconstructed from the current densities $\sigma \nabla u_i$. Thus, the rest of this subsection focuses only on the reconstruction of the vectors S_i.

We claim that the knowledge of the matrix H determines the matrix

$$(10.8) \qquad S = \left[S_1 \cdots S_d \right]$$

in $\widetilde{\Omega}$ up to a $SO(d, \mathbb{R})$-valued function. Indeed, first note that H is positive definite in $\widetilde{\Omega}$, as $H_{ij} = S_i \cdot S_j$ and $\{S_i\}_i$ is a basis of $\mathbb{R}^d$ in $\widetilde{\Omega}$ by (10.7). (This is a standard property of Gram matrices, the matrices of scalar products of a family of vectors.) Since H is symmetric and positive definite, we can construct the inverse of its square root

$$T = H^{-\frac{1}{2}} \quad \text{in } \widetilde{\Omega}.$$

Note that T is symmetric and positive definite. Thus, writing $R = ST$, we have that ${}^T RR = T^T S ST = THT = I$ in $\widetilde{\Omega}$ and $\det R > 0$ by (10.7), therefore R belongs to $SO(d, \mathbb{R})$. We obtain

$$(10.9) \qquad S = R H^{\frac{1}{2}} \quad \text{in } \widetilde{\Omega},$$

namely S is known in $\widetilde{\Omega}$ up to a (varying) rotation. It remains to determine the matrix $R(x)$ for every $x \in \widetilde{\Omega}$.

Write $T = (t_{ij})_{ij}$ and $T^{-1} = (t^{ij})_{ij}$ and set

(10.10) $$V_{ij} = (\nabla t_{ik})t^{kj}, \quad 1 \le i, j \le d.$$

The vector fields V_{ij} are known in $\widetilde{\Omega}$ and will be used for the reconstruction of R, which can be achieved by solving a PDE satisfied by R. This PDE can be best expressed in terms of a suitable parametrisation of the rotation matrix R. Thus, since the natural parametrisation of $\mathrm{SO}(d, \mathbb{R})$ depends on d, we shall study the two and three-dimensional cases separately.

10.2.1.1. *The two-dimensional case.* — Using the standard parametrisation of two-dimensional rotation matrices, it is convenient to write

$$R(x) = \begin{bmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{bmatrix}, \quad x \in \widetilde{\Omega}.$$

The angle θ satisfies a PDE, whose derivation is rather lengthy, and we have decided to omit it.

Proposition 10.5 (see [71], [40]). — *We have*

(10.11) $$\nabla\theta = \tfrac{1}{2}(V_{12} - V_{21}) + \tfrac{1}{4}\mathcal{J}\nabla\log\det H \quad in \ \widetilde{\Omega},$$

where $\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* V_{ij} *is defined in* (10.10).

It is now possible to reconstruct θ, thereby R, in $\widetilde{\Omega}$ up to a constant by integrating (10.11). The constant can be determined from the knowledge of $R(x_0)$. Once R is known, the vectors S_i can be reconstructed via (10.8) and (10.9).

In §10.2.2 we shall see how to use this local reconstruction algorithm to image σ in the whole domain Ω.

10.2.1.2. *The three-dimensional case.* — The three-dimensional case is slightly more involved. Indeed, while the Lie group $SO(2, \mathbb{R})$ is one dimensional, in dimension three the special orthogonal group $SO(3, \mathbb{R})$ is three-dimensional, and therefore obtaining the rotation matrix R is more complicated. As it was shown in [40], using the quaternion representation of the elements in $SO(3, \mathbb{R})$ yields an equation of the form

$$\nabla R = f(R, V_{ij}) \quad in \ \widetilde{\Omega},$$

for some function f that depends on R polynomially. As above, it is possible to integrate this PDE and reconstruct R, hence the vectors S_i, in $\widetilde{\Omega}$.

10.2.2. Global reconstruction. — The reconstruction procedure discussed in the previous subsection is based on estimate (10.7), which can be satisfied by using the results discussed in Chapters 6 and 7 in dimension two and three respectively.

10.2.2.1. *The two-dimensional case.* — In dimension two it is possible to satisfy (10.7) globally, this greatly simplifies the reconstruction. For simplicity, suppose that the domain Ω is convex.

Choose the two boundary values defined by

$$\varphi_1 = x_1, \quad \varphi_2 = x_2.$$

By Corollary 6.8 (see also Remark 6.4) we have

$$\det \begin{bmatrix} \nabla u^1 & \nabla u^2 \end{bmatrix}(x) \geq C, \quad x \in \overline{\Omega}.$$

Therefore, assuming that $\sigma(x_0)$, $\nabla u^1(x_0)$ and $\nabla u^2(x_0)$ are known for some $x_0 \in \partial\Omega$ by boundary measurements, it is possible to reconstruct σ in Ω by using the method discussed in §10.2.1.

10.2.2.2. *The three-dimensional case.* — In three dimensions it is possible to satisfy (10.7) only locally, by using the techniques discussed in Chapter 7, namely the complex geometric optics solutions and the Runge approximation. For brevity, we discuss only the approach based on CGO (Theorem 7.2); the approach based on the Runge approximation is similar (see Theorem 7.11).

Assume that $\sigma \in H^{\frac{3}{2}+3+\delta}(\mathbb{R}^3)$ for some $\delta > 0$. In view of Theorem 7.2, there exist boundary conditions $\varphi_1, \ldots, \varphi_4 \in C^2(\overline{\Omega}; \mathbb{R})$ such that

$$\left| \det \begin{bmatrix} \nabla u^1 \nabla u^2 \nabla u^3 \end{bmatrix}(x) \right| + \left| \det \begin{bmatrix} \nabla u^1 \nabla u^2 \nabla u^4 \end{bmatrix}(x) \right| > 0, \quad x \in \overline{\Omega}.$$

Therefore we have the decomposition

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_\ell = \{x \in \overline{\Omega} : \left| \det \begin{bmatrix} \nabla u^1 \nabla u^2 \nabla u^{2+\ell} \end{bmatrix}(x) \right| > 0\}.$$

Considering now the connected components Ω_j^ℓ ($l \in I_j$) of Ω_j, we have

$$\overline{\Omega} = \bigcup_{\substack{j=1,2 \\ \ell \in I_j}} \Omega_j^\ell.$$

Note that, by the compactness of $\overline{\Omega}$, we can assume that this union is finite.

It is now possible to apply the local reconstruction procedure discussed above as follows. Take $x_0 \in \partial\Omega$ such that $\sigma(x_0)$ and $\nabla u^i(x_0)$ are known for all $i = 1, \dots, 4$ by boundary measurements. Let $j_0 \in \{1, 2\}$ and $\ell_0 \in I_{j_0}$ be such that $x_0 \in \Omega_{j_0}^{\ell_0}$. By using the local reconstruction algorithm with measurements corresponding to the boundary values φ_1, φ_2 and $\varphi_{2+\ell_0}$ it is possible to reconstruct σ and ∇u^i in $\Omega_{j_0}^{\ell_0}$ for all i. Proceeding in the same way, we reconstruct σ and ∇u^i in Ω_j^ℓ for all j and l such that $\Omega_j^\ell \cap \Omega_{j_0}^{\ell_0} \neq \varnothing$. Repeating this argument a finite number of step, we can reconstruct σ uniquely in the whole $\overline{\Omega}$.

10.3. Thermoacoustic tomography

This section is devoted to the study of the quantitative step in thermoacoustic tomography (see Section 9.3). The problem of quantitative thermoacoustic tomography (QTAT) consists of the reconstruction of the conductivity σ from the knowledge of the internal data

$$H(x) = \sigma(x)\big|u(x)\big|^2, \quad x \in \Omega,$$

where the electric field u satisfies the Dirichlet boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u + (\omega^2 + i\omega\sigma)u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

This problem has attracted considerable attention in the past few years. An iterative algorithm for the reconstruction of σ is proposed in [216]. Uniqueness and stability for the problem, under certain assumptions on the parameter is shown in [41]. In this paper, the convergence of an iterative algorithm based

on the knowledge of multiple measurements $\sigma(x)|u^i(x)|^2$ is established, provided that the corresponding illuminations φ_i are suitably chosen. In [51] a general theory for the reconstruction of parameters of elliptic PDEs from the knowledge of their solutions is developed, and can be applied to our problem as in [38]. Another formulation of the same ideas is given in [28], where the authors exhibit an explicit reconstruction formula, namely an algebraic identity where σ is given explicitly from the data H. The formula uses multiple measurements where the illuminations φ_i are with suitably chosen.

Such a suitable choice of illumination can be achieved making use of the methods introduced in Chapters 7 and 8. This is the focus of this section.

Let us first give a precise formulation of the problem of quantitative thermoacoustics with multiple measurements. Let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a $C^{1,\alpha}$ bounded domain and $\sigma \in L^\infty(\Omega; \mathbb{R})$ be the conductivity of the medium such that

$$\Lambda^{-1} \le \sigma \le \Lambda \quad \text{in } \Omega$$

for some $\Lambda > 0$. Given $d+1$ boundary values $\varphi_i \in C^{1,\alpha}(\overline{\Omega}; \mathbb{C})$ $(i = 1, \ldots, d+1)$, let $u^i_\omega \in H^2_{\text{loc}}(\Omega; \mathbb{C}) \cap C^1(\overline{\Omega}; \mathbb{C})$ be the unique solution to

(10.12)
$$\begin{cases} \Delta u^i_\omega + (\omega^2 + \mathrm{i}\omega\sigma)u^i_\omega = 0 & \text{in } \Omega, \\ u^i_\omega = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

For well-posedness and regularity properties of this problem see Lemmata 3.2 and 8.5. The polarisation formula yields for $i = 1, \ldots, d+1$

$$\sigma \overline{u^1_\omega} u^i_\omega = \tfrac{1}{2}\left(\sigma|u^1_\omega + u^i_\omega|^2 - \sigma|u^1_\omega|^2 - \sigma|u^i_\omega|^2\right)$$
$$+ \tfrac{1}{2}\mathrm{i}\left(\sigma|\mathrm{i}u^1_\omega + u^i_\omega|^2 - \sigma|u^1_\omega|^2 - \sigma|u^i_\omega|^2\right).$$

All the factors on the right-hand side are measurable quantities. Indeed, by the linearity of (10.12) it is sufficient to use the boundary values φ_1, φ_i, $\varphi_1 + \varphi_i$ and $\mathrm{i}\varphi_1 + \varphi_i$ in (10.12) and measure the corresponding internal data $H(x) = \sigma(x)|u(x)|^2$. As a result, the quantities

$$H^i_\omega(x) = \sigma(x)\,\overline{u^1_\omega}(x)\,u^i_\omega(x), \quad x \in \Omega$$

can be considered as known data, and the conductivity σ has to be reconstructed.

10.3.1. Local reconstruction. — Let us first study the local reconstruction of σ from the knowledge of $d+1$ measurements $u_\omega^1, \ldots, u_\omega^{d+1}$, corresponding to a fixed frequency ω and $d+1$ boundary values $\varphi_1, \ldots, \varphi_{d+1}$. Let $\widetilde{\Omega} \subseteq \Omega$ be a subdomain in which the following constraints hold true:

$$(10.13\text{a}) \qquad |u_\omega^1| \geq C \quad \text{in } \widetilde{\Omega},$$

$$(10.13\text{b}) \qquad \left| \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix} \right| \geq C \quad \text{in } \widetilde{\Omega}.$$

We shall comment on how to satisfy these constraints globally in the following subsection. We now show that σ can be uniquely reconstructed in $\widetilde{\Omega}$.

We use the notation

$$v_\omega^i = \frac{H_\omega^i}{H_\omega^1} = \frac{u_\omega^i}{u_\omega^1},$$

so that v_ω^i is a known quantity as well. Note that v_ω^i is well-defined in $\widetilde{\Omega}$ since $u_\omega^1 \neq 0$ in $\widetilde{\Omega}$ by (10.13a). In view of (10.12) we have for $i = 2, \ldots, d+1$

$$-\operatorname{div}\big((u_\omega^1)^2 \nabla v_\omega^i \big) = -\operatorname{div}(u_\omega^1 \nabla u_\omega^i - u_\omega^i \nabla u_\omega^1) = u_\omega^i \Delta u_\omega^1 - u_\omega^1 \Delta u_\omega^i$$

$$= -u_\omega^i (\omega^2 + i\omega\sigma) u_\omega^1 + u_\omega^1 (\omega^2 + i\omega\sigma) u_\omega^i$$

$$= 0$$

in $\widetilde{\Omega}$. Recalling that $u_\omega^1, v_\omega^i \in H^2(\widetilde{\Omega}; \mathbb{C})$, expanding the left-hand side of this identity yields $\nabla (u_\omega^1)^2 \cdot \nabla v_\omega^i = -(u_\omega^1)^2 \Delta v_\omega^i$, whence $2\nabla u_\omega^1 \cdot \nabla v_\omega^i = -u_\omega^1 \Delta v_\omega^i$ in $\widetilde{\Omega}$ for $i = 2, \ldots, d+1$. Writing these equations in a more compact form we obtain

$$2\nabla u_\omega^1 \big[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1} \big] = -u_\omega^1 \big[\Delta v_\omega^2 \cdots \Delta v_\omega^{d+1} \big] \quad \text{in } \widetilde{\Omega}.$$

We would like to invert the matrix in the left-hand side in order to have an explicit expression for ∇u_ω^1. This is possible thanks to the identity

$$(10.14) \qquad \det \big[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1} \big] = (u_\omega^1)^{-(d+1)} \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix},$$

whose proof is trivial but rather lengthy, and has therefore been omitted. As a result, in view of (10.13b) the matrix $[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1}]$ is invertible in $\widetilde{\Omega}$ and so we obtain the following reconstruction formula.

Proposition 10.6. — *Under the above assumptions, we have*

$$2\nabla u_\omega^1 = -u_\omega^1 \big[\Delta v_\omega^2 \cdots \Delta v_\omega^{d+1} \big] \big[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1} \big]^{-1} \quad \text{in } \widetilde{\Omega}.$$

Since the v_ω^is are known, this first order PDE with unknown u_ω^1 can now be integrated in $\widetilde\Omega$, and u_ω^1 is determined uniquely in $\widetilde\Omega$ up to a complex multiplicative constant. In other words, we have $u_\omega^1 = cf$ for some unknown $c \in \mathbb{C}^*$ and some known function $f \in H^2(\widetilde\Omega; \mathbb{C})$. Finally, by using (10.12) we have

$$\omega^2 + \mathrm{i}\,\omega\sigma = -\frac{\Delta u_\omega^1}{u_\omega^1} = -\frac{\Delta f}{f} \quad \text{in } \widetilde\Omega,$$

whence σ can be uniquely reconstructed in $\widetilde\Omega$ through the explicit formula

$$\sigma = \mathrm{i}\,\omega^{-1}\left(\frac{\Delta f}{f} + \omega^2\right) \quad \text{in } \widetilde\Omega.$$

It is worth noting that, without additional regularity assumptions on σ, the above formula holds only almost everywhere in $\widetilde\Omega$.

Remark 10.7. — The above result allows a direct reconstruction of σ from the knowledge of the data H_i. Uniqueness and Lipschitz stability (with respect to appropriate Sobolev norms) for this inverse problem follow immediately from a careful inspection of the several steps involved. The stability constant depends on the lower bound $C > 0$ given in (10.13): the larger the better.

10.3.2. Global reconstruction. — We have seen above that σ can be reconstructed in a subdomain $\widetilde\Omega$ where the constraints given in (10.13) are satisfied for some $C > 0$. It remains to understand how to cover the whole domain Ω with several subdomains where the conditions (10.13) are satisfied for different boundary values and/or frequencies. As in the previous sections, since σ is unknown, this problem is highly non-trivial. In Chapters 7 and 8 we have discussed two methods to construct such boundary conditions in the case of a PDE with complex-valued coefficients, namely the complex geometric optics solutions and the multi-frequency approach. Also the Runge approximation approach could be used in this case, but we shall not discuss it since only the result for real-valued coefficients was proven in Chapter 7.

The use of CGO solutions allows us to satisfy (10.13) everywhere in the domain. In addition to the assumptions discussed above, we suppose that σ is the restriction to Ω of a function in $H^{d/2+1+\delta}(\mathbb{R}^d; \mathbb{R})$ for some $\delta > 0$. Condition (7.11) is satisfied by Lemma 8.5, so that all the assumptions of Theorem 7.3

are satisfied. By this result, for a fixed frequency $\omega > 0$ there exists an open set of boundary conditions $(\varphi_1, \ldots, \varphi_{d+1})$ in $C^2(\overline{\Omega}; \mathbb{C})$ such that

$$|u_\omega^1| \geq C \quad \text{in } \overline{\Omega}, \quad \left| \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix} \right| \geq C \quad \text{in } \overline{\Omega},$$

for some $C > 0$. As a consequence, the unknown conductivity σ can be uniquely and stably reconstructed in the whole domain Ω by using the method described in the previous subsection. As mentioned in Section 7.2, this approach has two main drawbacks: the high regularity required for σ and the non-explicit construction of the suitable boundary values φ_is.

It is possible to overcome these issues by using multiple measurements. More precisely, the multiple frequency approach discussed in Chapter 8 allows to choose *a priori* $d + 1$ real boundary values and several frequencies so that the desired constraints are satisfied everywhere in Ω for different frequencies, as we now describe. Let $\mathcal{A} = [K_{\min}, K_{\max}]$ be the microwave range of frequencies, for some $0 < K_{\min} < K_{\max}$, and define the finite set of frequencies $K^{(n)} \subseteq \mathcal{A}$ as in (8.8). Choose the $d + 1$ real boundary values defined by

$$\varphi_1 = 1, \quad \varphi_2 = x_1, \quad \ldots \quad , \varphi_{d+1} = x_d.$$

By Theorem 8.2, there exist a positive constant $C > 0$ and a number of frequencies $n \geq 2$ depending only on Ω, Λ and $\mathcal{A}$ and an open cover

$$\overline{\Omega} = \bigcup_{\omega \in K^{(n)}} \Omega_\omega$$

such that for every $\omega \in K^{(n)}$ we have

$$|u_\omega^1| \geq C \quad \text{in } \Omega_\omega, \quad \left| \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix} \right| \geq C \quad \text{in } \Omega_\omega.$$

Note that, since the functions H_ω^i are known, the domains Ω_ω are known. The unknown conductivity σ can then be obtained in each subdomain Ω_ω by using the local reconstruction method discussed above, with the measurements relative to the frequency ω. Since these subdomains cover the whole domain Ω, σ can be reconstructed globally. It is worth observing that in order to have explicitly constructed boundary conditions there is a price to pay: several measurements, corresponding to several frequencies ω, have to be taken.

10.4. Dynamic elastography

The quantitative step of dynamic elastography consists of the reconstruction of the tissue parameters from the knowledge of the tissue displacement. Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,\alpha}$ bounded domain, with $d = 2$ or $d = 3$. We consider here a simplified scalar elastic model, given by the Helmholtz equation

$$(10.15) \qquad \begin{cases} -\operatorname{div}(\mu \nabla u_\omega^i) - \rho\omega^2 u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

In this model $u_\omega^i \in C^1(\overline{\Omega}; \mathbb{R})$ represents (one component of) the tissue displacement. The shear modulus is $\mu \in W^{1,2}(\Omega; \mathbb{R}) \cap C^{0,\alpha}(\overline{\Omega}; \mathbb{R}_+)$, the density is $\rho \in L^\infty(\Omega; \mathbb{R}_+)$, the frequency is $\omega \in \mathbb{R}_+$, and $\varphi_i \in C^{1,\alpha}(\overline{\Omega}; \mathbb{R})$ is the boundary displacement. We assume that $\Lambda^{-1} \leq \mu, \rho \leq \Lambda$ almost everywhere in Ω for some $\Lambda > 0$. The inverse problem discussed in this section consists of the reconstruction of μ and ρ from the knowledge of several measurements u_ω^i, corresponding to several boundary conditions φ_i and possibly several frequencies ω.

The general problem of quantitative elastography modelled by the full Lamé system has attracted considerable attention over the last decade, see *e.g.* [154], [52], [131], [39]. The anisotropic case is treated in [45]. Given the knowledge of different full displacement vector fields, it is possible to uniquely and stably recover the Lamé parameters through explicit reconstruction algorithms. The scalar approximation of the linear system of elasticity was studied in [117] for the single-measurement case and in [38], [50], [51] for the multi-measurement case (see also [153]). The exposition presented below takes strong inspiration from these papers.

10.4.1. Local reconstruction. — Let us first study the local reconstruction of μ and ρ from the knowledge of $d + 1$ measurements $u_\omega^1, \dots u_\omega^{d+1}$, corresponding to a fixed frequency ω and $d+1$ boundary values $\varphi_1, \dots, \varphi_{d+1}$. Let $\widetilde{\Omega} \subseteq \Omega$ be a subdomain such that the following constraints hold true:

$$(10.16a) \qquad\qquad |u_\omega^1| \geq C \quad \text{in } \widetilde{\Omega},$$

$$(10.16b) \qquad\qquad \left| \det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix} \right| \geq C \quad \text{in } \widetilde{\Omega}.$$

We now show that μ and ρ can be uniquely reconstructed in $\widetilde{\Omega}$ provided that $\mu(x_0)$ is known at some point $x_0 \in \widetilde{\Omega}$. The inversion method is very similar to that discussed for quantitative thermoacoustic tomography in the previous section.

First, we observe that by (10.15) the quantities $v_\omega^i := u_\omega^i/u_\omega^1$ satisfy the PDE

$$-\operatorname{div}\left(\mu(u_\omega^1)^2\nabla v_\omega^i\right) = -\operatorname{div}(\mu u_\omega^1 \nabla u_\omega^i - \mu u_\omega^i \nabla u_\omega^1)$$
$$= u_\omega^i \operatorname{div}(\mu\nabla u_\omega^1) - u_\omega^1 \operatorname{div}(\mu\nabla u_\omega^i)$$
$$= -u_\omega^i\omega^2\rho u_\omega^1 + u_\omega^1\omega^2\rho u_\omega^i = 0$$

in $\widetilde{\Omega}$, for $i = 2,\ldots,d+1$. Arguing as in Lemma 10.1, we prove that u_ω^1, v_ω^i belongs to $H^2(\widetilde{\Omega};\mathbb{R})$. Expanding in turn the left-hand side of this identity yields

$$\nabla\mu \cdot (u_\omega^1)^2\nabla v_\omega^i = -\mu\operatorname{div}\left((u_\omega^1)^2\nabla v_\omega^i\right).$$

Recalling that μ and $|u_\omega^1|$ are strictly positive in $\widetilde{\Omega}$, this PDE can be rewritten as $\nabla(\log\mu) \cdot \nabla v_\omega^i = -(u_\omega^1)^{-2}\operatorname{div}((u_\omega^1)^2\nabla v_\omega^i)$, which in turn can be reformulated in a more compact form in $\widetilde{\Omega}$ as

$$\nabla(\log\mu)\left[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1}\right] = -(u_\omega^1)^{-2}\left[\operatorname{div}((u_\omega^1)^2\nabla v_\omega^2) \cdots \operatorname{div}\left((u_\omega^1)^2\nabla v_\omega^{d+1}\right)\right].$$

By (10.14) and (10.16b), the matrix $[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1}]$ is invertible in $\widetilde{\Omega}$, and so the above system of equations may be rewritten as follows.

Proposition 10.8. — *Under the above assumptions, we have in* $\widetilde{\Omega}$

$$\nabla(\log\mu) = -(u_\omega^1)^{-2}\left[\operatorname{div}\left((u_\omega^1)^2\nabla v_\omega^2\right) \cdots \operatorname{div}\left((u_\omega^1)^2\nabla v_\omega^{d+1}\right)\right]\left[\nabla v_\omega^2 \cdots \nabla v_\omega^{d+1}\right]^{-1}.$$

This first order PDE with unknown $\log\mu$ can be integrated in $\widetilde{\Omega}$, since u_ω^1 and the v_ω^is are known quantities, and μ can be uniquely reconstructed in $\widetilde{\Omega}$ up to a real multiplicative constant. However, since we assumed to know $\mu(x_0)$ for some $x_0 \in \widetilde{\Omega}$, this constant is uniquely determined.

Finally, using again (10.15) we immediately obtain

$$\rho = -\frac{\operatorname{div}(\mu\nabla u_\omega^1)}{\omega^2 u_\omega^1} \quad \text{in } \widetilde{\Omega},$$

which is an explicit reconstruction formula for ρ in $\widetilde{\Omega}$, since μ and u_ω^1 are now known in $\widetilde{\Omega}$.

10.4.2. Global reconstruction. — In the previous subsection, we have described a simple method for the local recovery of μ and ρ. The success of this approach relies on the fact that the constraints given in (10.16) are satisfied. The solutions to (10.15) do not satisfy (10.16) in Ω in general: waves oscillate, and so zeros must occur. The methods used in Chapters 7 and 8 can be used to cover the whole domain Ω with several subdomains in which the required constraints are satisfied for different measurements. Any of these methods, CGO solutions, the Runge approximation and the multi-frequency approach (see Remark 8.4) is applicable to this context. As the former and the latter were used in the previous section, we implement the Runge approximation approach for this modality.

The local reconstruction method described above is not fully local; namely, it requires the knowledge of μ in some point of the subdomain. It is unlikely that μ would be known *a priori* at one point of each subdomain. It is more relevant to assume that μ is known at a single locus $x_0 \in \Omega$, possibly near the boundary. This makes it impossible to readily use the global reconstruction method discussed in §10.3.2 for a very similar problem, and we will have to adapt the technique applied in §10.2.2. The idea is to start to apply the local reconstruction in the subdomain of Ω containing x_0, and then move to cover the whole domain using that subdomains overlap.

We also assume that $\mu \in C^{0,1}(\overline{\Omega}; \mathbb{R})$ (this hypothesis may be reduced to $C^{0,\alpha}$ in 2D and to $C^{0,\alpha} \cap W^{1,3}$ in 3D by Remark 7.6) and let $\Omega' \Subset \Omega$ be a connected subdomain such that $x_0 \in \Omega'$. Using the Runge approximation approach (Theorem 7.13), there exist $N \in \mathbb{N}^*$ depending only on Ω, Ω', Λ and $\|\mu\|_{C^{0,1}(\overline{\Omega};\mathbb{R})}$, $N \times (d+1)$ boundary values $\varphi_i^j \in H^{1/2}(\partial\Omega; \mathbb{R})$, $i = 1, \ldots, d+1$, $j = 1, \ldots, N$ and an open cover

$$\overline{\Omega'} = \bigcup_{j=1}^{N} \Omega_j$$

such that for every $j = 1, \ldots, N$ we have

$$(10.17) \qquad |u_\omega^{1,j}| \geq \tfrac{1}{2} \quad \text{in } \Omega_j, \quad \left| \det \begin{bmatrix} u_\omega^{1,j} & \cdots & u_\omega^{d+1,j} \\ \nabla u_\omega^{1,j} & \cdots & \nabla u_\omega^{d+1,j} \end{bmatrix} \right| \geq \tfrac{1}{2} \quad \text{in } \Omega_j,$$

where $u_\omega^{i,j}$ is the solution to (10.15) with boundary condition φ_i^j. Note that, as discussed in Section 7.3, the boundary conditions φ_i^j are not explicitly constructed. This would not be the case if the multi-frequency approach were used instead, provided for example that $\mu \equiv 1$ or $d = 2$.

As in §10.2.2, consider now the connected components Ω_j^ℓ ($\ell \in I_j$) of Ω_j. By construction we have

$$\overline{\Omega'} = \bigcup_{\substack{j=1,\ldots,N \\ \ell \in I_j}} \Omega_j^\ell.$$

Note that, by the compactness of $\overline{\Omega'}$, we may assume that this union is finite.

It is now possible to apply the local reconstruction procedure discussed in the previous subsection. Recall that $x_0 \in \Omega'$ is such that $\mu(x_0)$ is known. Let $j_0 \in \{1, \ldots, N\}$ and $\ell_0 \in I_{j_0}$ be such that $x_0 \in \Omega_{j_0}^{\ell_0}$. By (10.17), the required constraints (10.16) are satisfied in $\Omega_{j_0}^{\ell_0}$ for the boundary values $\varphi_1^{j_0}, \ldots, \varphi_{d+1}^{j_0}$. Using the local reconstruction algorithm we reconstruct μ and σ in $\Omega_{\ell_0}^{j_0}$. Iterating this approach, we reconstruct μ and σ in Ω_j^ℓ for all j and l such that $\Omega_j^\ell \cap \Omega_{\ell_0}^{j_0} \neq \varnothing$, using the boundary values $\varphi_1^j, \ldots, \varphi_{d+1}^j$. After a finite number of steps, we have reconstructed μ and σ uniquely everywhere in $\overline{\Omega'}$.

10.5. Photoacoustic tomography

The coupled step of photoacoustic tomography delivers an absorbed energy H inside the domain which has the form

$$H(x) = \Gamma(x)\,\mu(x)u(x), \quad x \in \Omega,$$

where Γ is the Grüneisen parameter, μ is the light absorption and u is the light intensity. While Γ and μ are properties of the tissue under consideration, u is the light field injected to obtain the measurements. The problem of quantitative photoacoustic tomography (QPAT) is the reconstruction of μ from the knowledge of (several measurements of) the internal energy H. In many cases, the condition $\Gamma = 1$ is assumed in order to make the problem more feasible.

This problem has attracted a lot of attention form the physical and mathematical communities, mainly over the past decade. We now mention the main

contribution on the mathematical aspects of this inverse problem; for further details, the reader is referred to the reviews [85], [98], [87]. Except for few exceptions, most methods are based on the partial differential equation satisfied by the light intensity u. Depending on the considered regime, light propagation is usually modelled by the radiative transport equation or, in case of highly scattering media, by its diffusion approximation [35]. The diffusion approximation has been considered in most contributions, see *e.g.* [18], [49] for the single-measurement case, [88], [46], [38], [186], [51] for the multi-measurement case and [86], [84] for the discussion of possible iterative methods. The full radiative transport equation is considered in [44], [88], [191].

Without adding further constraints, the above problem in its full generality is unsolvable. It was known that it is impossible to recover all the parameters in the diffusive regime, even with an infinite number of illuminations [46]. In order to overcome this issue, multi-frequency approaches were proposed in [47], [183]. An alternative approach is to include additional *a priori* assumptions: the unknown parameters may be assumed piecewise constant or more generally sparse with respect to a suitable basis [188], [169], [9], [151], [57].

It should be mentioned that all the methods discussed above regard the inversion in PAT as a two-step process: in the coupled step, the energy H is constructed, and in the quantitative step the unknowns have to be extracted. Very recently, one step methods have been developed, in which the intermediate step is skipped, and the unknown parameters are directly reconstructed from the acoustic measurements [112], [90], [182]

10.5.1. Reconstruction algorithm. — In this presentation, we are going to briefly present the method discussed in [46], [38] considering a very simple case: the diffusion approximation with constant scattering coefficient. Namely, assume that the light intensity u^i is the solution to

$$(10.18) \qquad \begin{cases} -\Delta u^i + \mu u^i = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ is a smooth bounded domain, $d = 2, 3$, $\mu \in L^\infty(\Omega; \mathbb{R})$ is such that $\Lambda^{-1} \leq \mu \leq \Lambda$ almost everywhere in Ω and $\varphi_i \in C^2(\overline{\Omega}; \mathbb{R}_+)$

for $i = 1, \ldots, d+1$. We need to reconstruct μ from the internal measurements

$$H_i(x) = \Gamma(x)\mu(x)u_i(x), \quad x \in \Omega.$$

As in the other hybrid modalities, non-zero constraints have to be satisfied. Let $\widetilde{\Omega} \subseteq \Omega$ be a subdomain such that

$$(10.19) \qquad \left| \det \begin{bmatrix} u^1 & \cdots & u^{d+1} \\ \nabla u^1 & \cdots & \nabla u^{d+1} \end{bmatrix} \right| \geq C \quad \text{in } \widetilde{\Omega}.$$

By the maximum principle, we immediately have that $u^1 \geq C$ in Ω for some $C > 0$, provided that $\varphi_1 > 0$ on $\partial\Omega$. Proceeding as in Sections 10.3 and 10.4, consider the known quantities

$$v^i = \frac{H_i}{H_1} = \frac{u^i}{u^1}.$$

It is immediate to see that $-\operatorname{div}((u^1)^2 \nabla v^i) = 0$ in Ω for every $i = 2, \ldots, d+1$. Arguing as in §10.3.1, thanks to (10.19) we can reconstruct u^1 in $\widetilde{\Omega}$ up to a multiplicative constant c. Finally, μ can be reconstructed in $\widetilde{\Omega}$ via

$$\mu = \frac{-\Delta(cu^1)}{cu^1}.$$

The above algorithm can be applied locally in each subdomain where (10.19) is satisfied for well chosen boundary values. The techniques presented in Chapter 7 can be used to cover the whole domain Ω with subdomains where the required constraint is satisfied for appropriate choices of the boundary conditions. The arguments are similar to those discussed in the previous sections, and have been omitted. The multi-frequency approach discussed in Chapter 8 is not applicable here, since (10.18) is independent of the frequency.

It is worth noting that, in two dimensions, condition (10.19) can be easily satisfied globally by using the theory developed in Chapter 6. Supposing that Ω is convex, it is enough to choose

$$\varphi_1 = 1, \quad \varphi_2 = x_1, \quad \varphi_3 = x_2.$$

Indeed, this implies that v^i is a solution of

$$\begin{cases} -\operatorname{div}((u^1)^2 \nabla v^i) = 0 & \text{in } \Omega, \\ v^i = x_{i-1} & \text{on } \partial\Omega, \end{cases}$$

for $i = 1, 2$. Therefore, for a fixed $\Omega' \Subset \Omega$, by Corollary 6.8 there exists $C > 0$ such that

$$\left| \det \left[\nabla v^2(x) \ \nabla v^2(x) \right] \right| \geq C, \quad x \in \Omega'.$$

By (10.14), this inequality is equivalent to (10.19).

BIBLIOGRAPHY

[1] S. Acosta & C. Montalto – "Multiwave imaging in an enclosure with variable wave speed", *Inverse Problems* **31** (2015), no. 6, p. 065009.

[2] L. V. Ahlfors – *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[3] G. S. Alberti – "On multiple frequency power density measurements", *Inverse Problems* **29** (2013), no. 11, p. 115007.

[4] _____ , "On local constraints and regularity of PDE in electromagnetics. Applications to hybrid imaging inverse problems", Ph. D. Thesis, University of Oxford, 2014.

[5] _____ , "Enforcing Local Non-Zero Constraints in PDEs and Applications to Hybrid Imaging Problems", *Comm. Partial Differential Equations* **40** (2015), no. 10, p. 1855–1883.

[6] _____ , "On multiple frequency power density measurements II. The full Maxwell's equations", *J. Differential Equations* **258** (2015), no. 8, p. 2767–2793.

[7] _____ , "Absence of Critical Points of Solutions to the Helmholtz Equation in 3D", *Arch. Ration. Mech. Anal.* **222** (2016), no. 2, p. 879–894.

[8] ———, "Hölder regularity for Maxwell's equations under minimal assumptions on the coefficients", *ArXiv e-prints: 1604.03741* (2016).

[9] G. S. Alberti & H. Ammari – "Disjoint sparsity for signal separation and applications to hybrid inverse problems in medical imaging", *Appl. Comput. Harmon. Anal.* **42** (2017), no. 2, p. 319–349.

[10] G. S. Alberti, H. Ammari & K. Ruan – "Multi-frequency acousto-electromagnetic tomography", in *A Panorama of Mathematics: Pure and Applied*, Contemp. Math., vol. 658, Amer. Math. Soc., Providence, RI, 2016, p. 67–79.

[11] G. S. Alberti, G. Bal & M. Di Cristo – "Critical points for elliptic equations with prescribed boundary conditions", *Arch. Ration. Mech. Anal.* **226** (2017), no. 1, p. 117–141.

[12] G. S. Alberti & Y. Capdeboscq – "Elliptic Regularity Theory Applied to Time Harmonic Anisotropic Maxwell's Equations with Less than Lipschitz Complex Coefficients", *SIAM J. Math. Anal.* **46** (2014), no. 1, p. 998–1016.

[13] ———, "On local non-zero constraints in PDE with analytic coefficients", in *Imaging, Multi-scale and High Contrast Partial Differential Equations*, Contemp. Math., vol. 660, Amer. Math. Soc., Providence, RI, 2016, p. 89–97.

[14] G. Alessandrini & R. Magnanini – "Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions", *SIAM J. Math. Anal.* **25** (1994), no. 5, p. 1259–1268.

[15] G. Alessandrini – "An identification problem for an elliptic equation in two variables", *Ann. Mat. Pura Appl. (4)* **145** (1986), p. 265–295.

[16] ———, "Critical points of solutions of elliptic equations in two variables", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14** (1987), no. 2, p. 229–256 (1988).

[17] ———, "The length of level lines of solutions of elliptic equations in the plane", *Arch. Rational Mech. Anal.* **102** (1988), no. 2, p. 183–191.

[18] G. Alessandrini, M. Di Cristo, E. Francini & S. Vessella – "Stability for quantitative photoacoustic tomography with well-chosen illuminations", *Ann. Mat. Pura Appl. (4)* **196** (2017), no. 2, p. 395–406.

[19] G. Alessandrini & V. Nesi – "Univalent σ-harmonic mappings", *Arch. Ration. Mech. Anal.* **158** (2001), no. 2, p. 155–171.

[20] ———, "Quantitative estimates on Jacobians for hybrid inverse problems", *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"* **8** (2015), no. 3, p. 25–41.

[21] G. Alessandrini, L. Rondi, E. Rosset & S. Vessella – "The stability for the Cauchy problem for elliptic equations", *Inverse Problems* **25** (2009), no. 12, p. 123004.

[22] G. Allaire – "Homogenization and two-scale convergence", *SIAM J. Math. Anal.* **23** (1992), no. 6, p. 1482–1518.

[23] A. Ammari, M. S. Vogelius & D. Volkov – "Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations.", *J. Math. Pures Appl.* **80** (2001), p. 769–814.

[24] H. Ammari – *An introduction to mathematics of emerging biomedical imaging*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 62, Springer, Berlin, 2008.

[25] H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter & M. Fink – "Electrical impedance tomography by elastic deformation", *SIAM J. Appl. Math.* **68** (2008), no. 6, p. 1557–1573.

[26] H. Ammari, E. Bossy, V. Jugnon & H. Kang – "Mathematical modeling in photoacoustic imaging of small absorbers", *SIAM Rev.* **52** (2010), no. 4, p. 677–695.

[27] H. Ammari, Y. Capdeboscq, F. de Gournay, A. Rozanova-Pierrat & F. Triki – "Microwave imaging by elastic deformation", *SIAM J. Appl. Math.* **71** (2011), no. 6, p. 2112–2130.

[28] H. Ammari, Garnier, W. Jing & L. H. Nguyen – "Quantitative thermoacoustic imaging: An exact reconstruction formula", *J. Differential Equations* **254** (2013), no. 3, p. 1375–1395.

[29] H. Ammari, J. Garnier, H. Kang, L. H. Nguyen & L. Seppecher – "Mathematics of super-resolution biomedical imaging", Tech. Report 2016-31, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2016.

[30] H. Ammari & H. Kang – *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics, vol. 1846, Springer, 2004.

[31] _____ , *Polarization and Moment Tensors With Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences, vol. 162, Springer, 2007.

[32] H. Ammari, L. Giovangigli, L. H. Nguyen & J.-K. Seo – "Admittivity imaging from multi-frequency micro-electrical impedance tomography", *J. Math. Anal. Appl.* **449** (2017), no. 2, p. 1601–1618.

[33] C. Amrouche, C. Bernardi, M. Dauge & V. Girault – "Vector potentials in three-dimensional non-smooth domains", *Math. Methods Appl. Sci.* **21** (1998), no. 9, p. 823–864.

[34] C. Amrouche & N. E. H. Seloula – "L^p-theory for vector potentials and Sobolev's inequalities for vector fields: application to the Stokes equations with pressure boundary conditions", *Math. Models Methods Appl. Sci.* **23** (2013), no. 1, p. 37–92.

[35] S. R. Arridge – "Optical tomography in medical imaging", *Inverse Problems* **15** (1999), no. 2, p. R41.

[36] K. Astala & L. Päivärinta – "Calderón's inverse conductivity problem in the plane", *Ann. of Math. (2)* **163** (2006), no. 1, p. 265–299.

[37] P. Auscher & M. Qafsaoui – "Observations on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients", *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **5** (2002), no. 2, p. 487–509.

[38] G. Bal – "Explicit Reconstructions in QPAT, QTAT, TE, and MRE", *ArXiv e-prints* (2012).

[39] G. Bal, C. Bellis, S. Imperiale & F. Monard – "Reconstruction of constitutive parameters in isotropic linear elasticity from noisy full-field measurements", *Inverse Problems* **30** (2014), no. 12, p. 125004.

[40] G. Bal, E. Bonnetier, F. Monard & F. Triki – "Inverse diffusion from knowledge of power densities", *Inverse Probl. Imaging* **7** (2013), no. 2, p. 353–375.

[41] G. Bal, K. Ren, G. Uhlmann & T. Zhou – "Quantitative thermoacoustics and related problems", *Inverse Problems* **27** (2011), no. 5, p. 055007.

[42] G. Bal – "Hybrid inverse problems and internal functionals", in *Inverse problems and applications: inside out. II*, Math. Sci. Res. Inst. Publ., vol. 60, Cambridge Univ. Press, Cambridge, 2013, p. 325–368.

[43] G. Bal, C. Guo & F. Monard – "Inverse anisotropic conductivity from internal current densities", *Inverse Problems* **30** (2014), no. 2, p. 025001.

[44] G. Bal, A. Jollivet & V. Jugnon – "Inverse transport theory of photoacoustics", *Inverse Problems* **26** (2010), no. 2, p. 025011.

[45] G. Bal, F. Monard & G. Uhlmann – "Reconstruction of a fully anisotropic elasticity tensor from knowledge of displacement fields", *SIAM J. Appl. Math.* **75** (2015), no. 5, p. 2214–2231.

[46] G. Bal & K. Ren – "Multi-source quantitative photoacoustic tomography in a diffusive regime", *Inverse Problems* **27** (2011), no. 7, p. 075003.

[47] _____ , "On multi-spectral quantitative photoacoustic tomography in diffusive regime", *Inverse Problems* **28** (2012), no. 2, p. 025010.

[48] G. Bal & J. C. Schotland – "Ultrasound-modulated bioluminescence tomography", *Phys. Rev. E* **89** (2014), p. 031201.

[49] G. Bal & G. Uhlmann – "Inverse diffusion theory of photoacoustics", *Inverse Problems* **26** (2010), no. 8, p. 085010.

[50] _____ , "Reconstructions for some coupled-physics inverse problems", *Appl. Math. Lett.* **25** (2012), no. 7, p. 1030–1033.

[51] ———, "Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions", *Comm. Pure Appl. Math.* **66** (2013), no. 10, p. 1629–1652.

[52] P. E. Barbone, C. E. Rivas, I. Harari, U. Albocher, A. A. Oberai & Y. Zhang – "Adjoint-weighted variational formulation for the direct solution of inverse problems of general linear elasticity with full interior data", *Internat. J. Numer. Methods Engrg.* **81** (2010), no. 13, p. 1713–1736.

[53] C. Bardos, G. Lebeau & J. Rauch – "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30** (1992), no. 5, p. 1024–1065.

[54] P. Bauman, A. Marini & V. Nesi – "Univalent solutions of an elliptic system of partial differential equations arising in homogenization", *Indiana Univ. Math. J.* **50** (2001), no. 2, p. 747–757.

[55] M. F. Ben Hassen & E. Bonnetier – "An asymptotic formula for the voltage potential in a perturbed ε-periodic composite medium containing misplaced inclusions of size ε", *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), no. 4, p. 669–700.

[56] A. Bensoussan, J.-L. Lions & G. C. Papanicolaou – *Asymptotic analysis for periodic structures*, North-Holland Publishing Co., Amsterdam, 1978.

[57] E. Beretta, M. Muszkieta, W. Naetar & O. Scherzer – "A variational method for quantitative photoacoustic tomography with piecewise constant coefficients", in *Variational Methods: In Imaging and Geometric Control*, Radon Series on Computational and Applied Mathematics, vol. 18, De Gruyter, 2017, p. 202–224.

[58] L. Bers – "Local behavior of solutions of general linear elliptic equations", *Comm. Pure Appl. Math.* **8** (1955), p. 473–496.

[59] E. Bonnetier & F. Triki – "Asymptotics in the presence of inclusions of small volume for a conduction equation: a case with a non-smooth reference potential", in *Imaging microstructures*, Contemp. Math., vol. 494, Amer. Math. Soc., Providence, RI, 2009, p. 95–111.

[60] T. Bowen – "Radiation-Induced Thermoacoustic Soft-Tissue Imaging", *IEEE Trans. on Sonics and Ultrasonics* **29** (1982), no. 3, p. 187.

[61] B. V. Boyarskiĭ – "Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients", *Mat. Sb. N.S.* **43(85)** (1957), p. 451–503.

[62] H. Brezis & L. Nirenberg – "Degree theory and bmo; part i: Compact manifolds without boundaries", *Selecta Mathematica* **1** (1995), no. 2, p. 197–263 (English).

[63] M. Briane & G. W. Milton – "Homogenization of the three-dimensional Hall effect and change of sign of the Hall coefficient", *Arch. Ration. Mech. Anal.* **193** (2009), no. 3, p. 715–736.

[64] M. Briane, G. W. Milton & V. Nesi – "Change of sign of the corrector's determinant for homogenization in three-dimensional conductivity", *Arch. Ration. Mech. Anal.* **173** (2004), no. 1, p. 133–150.

[65] M. Briane, G. W. Milton & A. Treibergs – "Which electric fields are realizable in conducting materials?", *ESAIM: Mathematical Modelling and Numerical Analysis* **48** (2014), p. 307–323.

[66] R. M. Brown & R. H. Torres – "Uniqueness in the inverse conductivity problem for conductivities with $3/2$ derivatives in L^p, $p > 2n$", *J. Fourier Anal. Appl.* **9** (2003), no. 6, p. 563–574.

[67] M. Brühl, M. Hanke & M. S. Vogelius – "A direct impedance tomography algorithm for locating small inhomogeneities", *Numer. Math.* **93** (2003), no. 4, p. 635–654.

[68] N. Burq & P. Gérard – "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997), no. 7, p. 749–752.

[69] L. A. Caffarelli & A. Friedman – "Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations", *J. Differential Equations* **60** (1985), no. 3, p. 420–433.

[70] A.-P. Calderón – "On an inverse boundary value problem", in *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, Soc. Brasil. Mat., Rio de Janeiro, 1980, p. 65–73.

[71] Y. Capdeboscq, J. Fehrenbach, F. de Gournay & O. Kavian – "Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements", *SIAM J. Imaging Sci.* **2** (2009), no. 4, p. 1003–1030.

[72] Y. Capdeboscq, G. Leadbetter & A. Parker – "On the scattered field generated by a ball inhomogeneity of constant index in dimension three", in *Multi-scale and high-contrast PDE: from modelling, to mathematical analysis, to inversion*, Contemp. Math., vol. 577, Amer. Math. Soc., Providence, RI, 2012, p. 61–80.

[73] Y. Capdeboscq & M. S. Vogelius – "A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction", *M2AN Math. Model. Numer. Anal.* **37** (2003), no. 1, p. 159–173.

[74] _____ , "Pointwise polarization tensor bounds, and applications to voltage perturbations caused by thin inhomogeneities", *Asymptot. Anal.* **50** (2006), no. 3-4, p. 175–204.

[75] Y. Capdeboscq – "On the scattered field generated by a ball inhomogeneity of constant index", *Asymptot. Anal.* **77** (2012), no. 3-4, p. 197–246.

[76] _____ , "Corrigendum: On the scattered field generated by a ball inhomogeneity of constant index", *Asymptot. Anal.* **88** (2014), no. 3, p. 185–186.

[77] _____ , "On a counter-example to quantitative Jacobian bounds", *J. Éc. polytech. Math.* **2** (2015), p. 171–178.

[78] P. Caro & K. M. Rogers – "Global uniqueness for the Calderón problem with Lipschitz conductivities", *Forum Math. Pi* **4** (2016), p. e2, 28.

[79] D. J. Cedio-Fengya, S. Moskow & M. S. Vogelius – "Identification of conductivity imperfections of small diameter by boundary measurements. continuous dependence and computational reconstruction", *Inverse Problems* **14** (1998), p. 553–595.

[80] J. Chen & Y. Yang – "Inverse problem of electro-seismic conversion", *Inverse Problems* **29** (2013), no. 11, p. 115006.

[81] G. Choquet – "Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques", *Bull. Sci. Math. (2)* **69** (1945), p. 156–165.

[82] D. Colton & R. Kress – *Inverse acoustic and electromagnetic scattering theory*, second ed., Applied Mathematical Sciences, vol. 93, Springer-Verlag, Berlin, 1998.

[83] D. Colton & L. Päivärinta – "The uniqueness of a solution to an inverse scattering problem for electromagnetic waves", *Arch. Rational Mech. Anal.* **119** (1992), no. 1, p. 59–70.

[84] B. T. Cox, S. R. Arridge & P. C. Beard – "Photoacoustic tomography with a limited-aperture planar sensor and a reverberant cavity", *Inverse Problems* **23** (2007), no. 6, p. S95–S112.

[85] B. T. Cox, J. G. Laufer & P. C. Beard – "The challenges for quantitative photoacoustic imaging", *Proc. SPIE* **7177** (2009), p. 717713–717713–9.

[86] B. T. Cox, S. R. Arridge, K. P. Köstli & P. C. Beard – "Two-dimensional quantitative photoacoustic image reconstruction of absorption distributions in scattering media by use of a simple iterative method", *Appl. Opt.* **45** (2006), no. 8, p. 1866–1875.

[87] B. T. Cox, J. G. Laufer, S. R. Arridge & P. C. Beard – "Quantitative spectroscopic photoacoustic imaging: a review", *Journal of Biomedical Optics* **17** (2012), no. 6, p. 061202–1–061202–22.

[88] B. T. Cox, T. Tarvainen & S. R. Arridge – "Multiple illumination quantitative photoacoustic tomography using transport and diffusion models", in *Tomography and inverse transport theory*, Contemp. Math., vol. 559, Amer. Math. Soc., Providence, RI, 2011, p. 1–12.

[89] R. Dautray & J.-L. Lions – *Mathematical analysis and numerical methods for science and technology*, Springer Verlag, Berlin, 1993.

[90] T. Ding, R. Kui & S. Valléfeblian – "A one-step reconstruction algorithm for quantitative photoacoustic imaging", *Inverse Problems* **31** (2015), no. 9, p. 095005.

[91] P. Duren – *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, vol. 156, Cambridge University Press, Cambridge, 2004.

[92] T. Duyckaerts, X. Zhang & E. Zuazua – "On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials", *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* **25** (2008), no. 1, p. 1–41.

[93] P. Elbau, L. Mindrinos & O. Scherzer – "Inverse problems of combined photoacoustic and optical coherence tomography", *Mathematical Methods in the Applied Sciences* **40** (2017), no. 3, p. 505–522, mma.3915.

[94] S. Ervedoza & E. Zuazua – "The wave equation: Control and numerics", in *Control of Partial Differential Equations*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2012, p. 245–339 (English).

[95] L. C. Evans – *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[96] A. Friedman & M. Vogelius – "Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence.", *Arch. Rat. Mech. Anal.* **105** (1989), p. 299–326.

[97] K. O. Friedrichs – "Differential forms on Riemannian manifolds", *Comm. Pure Appl. Math.* **8** (1955), p. 551–590.

[98] H. Gao, S. Osher & H. Zhao – "Quantitative photoacoustic tomography", in *Mathematical modeling in biomedical imaging. II*, Lecture Notes in Math., vol. 2035, Springer, Heidelberg, 2012, p. 131–158.

[99] B. Gebauer & O. Scherzer – "Impedance-acoustic tomography", *SIAM Journal on Applied Mathematics* **69** (2008), no. 2, p. 565–576.

[100] F. W. Gehring – "The L^p-integrability of the partial derivatives of a quasiconformal mapping", *Acta Math.* **130** (1973), p. 265–277.

[101] M. Gendron, R. Guardo & M. Bertrand – "Experimental setup for developing acousto-electric interaction imaging", in *Engineering in Medicine and Biology Society, 2009. EMBC 2009. Annual International Conference of the IEEE*, Sept 2009, p. 2279–2283.

[102] J.-L. Gennisson, T. Deffieux, M. Fink & M. Tanter – "Ultrasound elastography: Principles and techniques", *Diagnostic and Interventional Imaging* **94** (2013), no. 5, p. 487 – 495, Ultrasound elastography.

[103] M. Giaquinta & L. Martinazzi – *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, second ed., Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 11, Edizioni della Normale, Pisa, 2012.

[104] D. Gilbarg & N. S. Trudinger – *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Comprehensive Studies in Mathematics, Springer-Verlag, 1983.

[105] ———, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[106] V. Girault & P.-A. Raviart – *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986, Theory and algorithms.

[107] J. F. Greenleaf, M. Fatemi & M. Insana – "Selected methods for imaging elastic properties of biological tissues", *Annual Review of Biomedical Engineering* **5** (2003), no. 1, p. 57–78, PMID: 12704084.

[108] R. Griesmaier – "A general perturbation formula for electromagnetic fields in presence of low volume scatterers", *ESAIM Math. Model. Numer. Anal.* **45** (2011), no. 6, p. 1193–1218.

[109] C. W. Groetsch – *Inverse problems*, Mathematical Association of America, Washington, DC, 1999, Activities for undergraduates.

[110] B. Haberman & D. Tataru – "Uniqueness in Calderón's problem with Lipschitz conductivities", *Duke Math. J.* **162** (2013), no. 3, p. 496–516.

[111] B. Haberman – "Uniqueness in Calderón's problem for conductivities with unbounded gradient", *Comm. Math. Phys.* **340** (2015), no. 2, p. 639–659.

[112] M. Haltmeier, L. Neumann & S. Rabanser – "Single-stage reconstruction algorithm for quantitative photoacoustic tomography", *Inverse Problems* **31** (2015), no. 6, p. 065005.

[113] Q. Han, R. Hardt & F. Lin – "Geometric measure of singular sets of elliptic equations", *Comm. Pure Appl. Math.* **51** (1998), no. 11-12, p. 1425–1443.

[114] P. Hartman & A. Wintner – "On the local behavior of solutions of non-parabolic partial differential equations", *Amer. J. Math.* **75** (1953), p. 449–476.

[115] K. Hasanov, A. Ma, R. Yoon, A. Nachman & M. Joy – "A new approach to current density impedance imaging", in *Engineering in Medicine and Biology Society, 2004. IEMBS '04. 26th Annual International Conference of the IEEE*, vol. 1, Sept 2004, p. 1321–1324.

[116] B. Holman & L. Kunyansky – "Gradual time reversal in thermo- and photo-acoustic tomography within a resonant cavity", *Inverse Problems* **31** (2015), no. 3, p. 035008.

[117] N. Honda, J. McLaughlin & G. Nakamura – "Conditional stability for a single interior measurement", *Inverse Problems* **30** (2014), no. 5, p. 055001.

[118] T. Iwaniec – "The Gehring lemma", in *Quasiconformal mappings and analysis (Ann Arbor, MI, 1995)*, Springer, New York, 1998, p. 181–204.

[119] J. Jossinet, B. Lavandier & D. Cathignol – "The phenomenology of acousto-electric interaction signals in aqueous solutions of electrolytes", *Ultrasonics* **36** (1998), no. 1-5, p. 607 – 613, Ultrasonics International 1997.

[120] M. Kadic, R. Schittny, T. Bückmann, C. Kern & M. Wegener – "Hall-effect sign inversion in a realizable 3d metamaterial", *Phys. Rev. X* **5** (2015), p. 021030.

[121] S. Kaliski & W. Nowacki – "Excitation of Mechanical-electromagnetic Waves Induced by a Thermal Shock", *Bull Acad. Polonaise Sci. - Serie des Sciences et Techniques* **10** (1962), no. 1, p. 25–33.

[122] A. Kirsch & F. Hettlich – *The mathematical theory of time-harmonic Maxwell's equations*, Applied Mathematical Sciences, vol. 190, Springer, Cham, 2015, Expansion-, integral-, and variational methods.

[123] H. Kneser – "Lösung der Aufgabe 41", *Jahresber. Deutsch. Math.-Verein* **35** (1926), p. 123–124.

[124] I. Kocyigit – "Acousto-electric tomography and CGO solutions with internal data", *Inverse Problems* **28** (2012), no. 12, p. 125004.

[125] R. V. Kohn & M. S. Vogelius – "Determining conductivity by boundary measurements. II. Interior results", *Comm. Pure Appl. Math.* **38** (1985), no. 5, p. 643–667.

[126] V. Komornik – *Exact controllability and stabilization*, RAM: Research in Applied Mathematics, Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994, The multiplier method.

[127] P. Kuchment & L. Kunyansky – "Mathematics of Photoacoustic and Thermoacoustic Tomography", in *Handbook of Mathematical Methods in Imaging* (O. Scherzer, ed.), Springer New York, 2011, p. 817–865.

[128] P. Kuchment – "Mathematics of hybrid imaging: a brief review", in *The mathematical legacy of Leon Ehrenpreis*, Springer Proc. Math., vol. 16, Springer, Milan, 2012, p. 183–208.

[129] P. Kuchment & L. Kunyansky – "2D and 3D reconstructions in acousto-electric tomography", *Inverse Problems* **27** (2011), no. 5, p. 055013.

[130] L. Kunyansky, B. Holman & B. T. Cox – "Photoacoustic tomography in a rectangular reflecting cavity", *Inverse Problems* **29** (2013), no. 12, p. 125010.

[131] R.-Y. Lai – "Uniqueness and stability of Lamé parameters in elastography", *J. Spectr. Theory* **4** (2014), no. 4, p. 841–877.

[132] L. J. Landau – "Bessel Functions: Monotonicity and bounds", *J. London Math. Soc.,* *61* (2000), p. 197–215.

[133] I. Lasiecka & R. Triggiani – *Control theory for partial differential equations: continuous and approximation theories. II*, Encyclopedia of Mathematics and its Applications, vol. 75, Cambridge University Press, Cambridge, 2000, Abstract hyperbolic-like systems over a finite time horizon.

[134] R. S. Laugesen – "Injectivity can fail for higher-dimensional harmonic extensions", *Complex Variables Theory Appl.* **28** (1996), no. 4, p. 357–369.

[135] B. Lavandier, J. Jossinet & D. Cathignol – "Experimental measurement of the acousto-electric interaction signal in saline solution", *Ultrasonics* **38** (2000), no. 9, p. 929–936.

[136] P. D. Lax – "A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations", *Comm. Pure Appl. Math.* **9** (1956), p. 747–766.

[137] J.-Y. Lee – "A reconstruction formula and uniqueness of conductivity in MREIT using two internal current distributions", *Inverse Problems* **20** (2004), no. 3, p. 847–858.

[138] R. Leis – *Initial-boundary value problems in mathematical physics*, B. G. Teubner, Stuttgart, 1986.

[139] R. M. Lerner, K. J. Parker, J. Holen, R. Gramiak & R. C. Waag – "Sono-elasticity: Medical elasticity images derived from ultrasound signals in mechanically vibrated targets", in *Acoustical Imaging* (L. Kessler, ed.), Acoustical Imaging, vol. 16, Springer US, 1988, p. 317–327 (English).

[140] H. Lewy – "On the non-vanishing of the Jacobian in certain one-to-one mappings", *Bull. Amer. Math. Soc.* **42** (1936), no. 10, p. 689–692.

[141] Y. Y. Li & L. Nirenberg – "Estimates for elliptic systems from composite material", *Comm. Pure Appl. Math* **56** (2003), p. 892–925.

[142] Y. Y. Li & M. S. Vogelius – "Gradient estimates for solutions of divergence form elliptic equations with discontinuous coefficients", *Arch. Rat. Mech. Anal.* **153** (2000), p. 91–151.

[143] J. C. Lin & K. H. Chan – "Microwave Thermoelastic Tissue Imaging - System-Design", *IEEE Trans. Microw. Theory Tech.* **32** (1984), no. 8, p. 854–860.

[144] J.-L. Lions – *Some Methods in the Mathematical Analysis of Systems and their Controls*, Gordon and Breach, New York, 1981.

[145] ———, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research

in Applied Mathematics], vol. 8, Masson, Paris, 1988, Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[146] _____ , "Exact controllability, stabilization and perturbations for distributed systems", *SIAM Rev.* **30** (1988), no. 1, p. 1–68.

[147] R. Lipton – "Inequalities for electric and elastic polarization tensors with applications to random composites", *J. Mech. Phys. Solids* **41** (1993), no. 5, p. 809–833.

[148] R. Lipton & T. Mengesha – "Representation formulas for l^∞ norms of weakly convergent sequences of gradient fields in homogenization", *ESAIM: Mathematical Modelling and Numerical Analysis* **46** (2012), p. 1121–1146.

[149] F. Macià & E. Zuazua – "On the lack of observability for wave equations: a Gaussian beam approach", *Asymptot. Anal.* **32** (2002), no. 1, p. 1–26.

[150] B. Malgrange – "Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution", *Ann. Inst. Fourier, Grenoble* **6** (1955-1956), p. 271–355.

[151] E. Malone, S. Powell, B. T. Cox & S. Arridge – "Reconstruction-classification method for quantitative photoacoustic tomography", *Journal of Biomedical Optics* **20** (2015), no. 12, p. 126004.

[152] N. Mandache – "Exponential instability in an inverse problem for the Schrödinger equation", *Inverse Problems* **17** (2001), no. 5, p. 1435.

[153] J. McLaughlin, A. Oberai & J.-R. Yoon – "Formulas for detecting a spherical stiff inclusion from interior data: a sensitivity analysis for the Helmholtz equation", *Inverse Problems* **28** (2012), no. 8, p. 084004.

[154] J. R. McLaughlin, N. Zhang & A. Manduca – "Calculating tissue shear modules and pressure by 2D log-elastographic methods", *Inverse Problems* **26** (2010), no. 8, p. 085007.

[155] R. B. Melrose & J. Sjöstrand – "Singularities of boundary value problems. I", *Comm. Pure Appl. Math.* **31** (1978), no. 5, p. 593–617.

[156] _____, "Singularities of boundary value problems. II", *Comm. Pure Appl. Math.* **35** (1982), no. 2, p. 129–168.

[157] N. G. Meyers – "An L^p-estimate for the gradient of solutions of second order elliptic divergence equations", *Ann. Scuola Norm. Sup. Pisa (3)* **17** (1963), p. 189–206.

[158] G. W. Milton – *The theory of composites*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2002.

[159] A. Moiola & E. A. Spence – "Acoustic transmission problems: wavenumber-explicit bounds and resonance-free regions", *ArXiv e-prints: 1702.00745* (2017).

[160] F. Monard & G. Bal – "Inverse anisotropic diffusion from power density measurements in two dimensions", *Inverse Problems* **28** (2012), no. 8, p. 084001.

[161] _____, "Inverse diffusion problems with redundant internal information", *Inverse Probl. Imaging* **6** (2012), no. 2, p. 289–313.

[162] _____, "Inverse anisotropic conductivity from power densities in dimension $n \geq 3$", *Comm. Partial Differential Equations* **38** (2013), no. 7, p. 1183–1207.

[163] P. Monk – *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[164] A. B. Movchan – "Integral characteristics of elastic inclusions and cavities in the two-dimensional theory of elasticity", *European J. Appl. Math.* **3** (1992), no. 1, p. 21–30.

[165] A. B. Movchan & S. K. Serkov – "The Pólya-Szegő matrices in asymptotic models of dilute composites", *European J. Appl. Math.* **8** (1997), no. 6, p. 595–621.

[166] F. Murat & L. Tartar – "Calcul des variations et homogénéisation", in *Les Méthodes de l'Homogénéisation Théorie et Applications en Physique*, Coll. Dir. Etudes et Recherches EDF, Eyrolles, 1985, p. 319–369.

[167] _____ , "*H*-convergence", in *Topics in the mathematical modelling of composite materials*, Progr. Nonlinear Differential Equations Appl., vol. 31, Birkhäuser Boston, Boston, MA, 1997, p. 21–43.

[168] R. MUTHUPILLAI, D. LOMAS, P. ROSSMAN, J. GREENLEAF, A. MANDUCA & R. EHMAN – "Magnetic resonance elastography by direct visualization of propagating acoustic strain waves", *Science* **269** (1995), no. 5232, p. 1854–1857.

[169] W. NAETAR & O. SCHERZER – "Quantitative photoacoustic tomography with piecewise constant material parameters", *SIAM J. Imaging Sci.* **7** (2014), no. 3, p. 1755–1774.

[170] J.-C. NÉDÉLEC – *Acoustic and electromagnetic equations*, Applied Mathematical Sciences, vol. 144, Springer-Verlag, New York, 2001, Integral representations for harmonic problems.

[171] H.-M. NGUYEN & M. S. VOGELIUS – "A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), no. 6, p. 2283–2315.

[172] _____ , "Full range scattering estimates and their application to cloaking", *Arch. Ration. Mech. Anal.* **203** (2012), no. 3, p. 769–807.

[173] L. NIRENBERG – "Uniqueness in Cauchy problems for differential equations with constant leading coefficients", *Comm. Pure Appl. Math.* **10** (1957), p. 89–105.

[174] W. NOWACKI – *Dynamic problems of thermoelasticity*, Noordhoff/Kluwer, 1975.

[175] R. OLAFSSON, R. WITTE, S.-W. HUANG & M. O'DONNELL – "Ultrasound current source density imaging", *Biomedical Engineering, IEEE Transactions on* **55** (2008), no. 7, p. 1840–1848.

[176] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT & C. W. CLARK (eds.) – *NIST handbook of mathematical functions*, U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010, With 1 CD-ROM (Windows, Macintosh and UNIX).

[177] J. Ophir, I. Céspedes, H. Ponnekanti, Y. Yazdi & X. Li – "Elastography: A quantitative method for imaging the elasticity of biological tissues", *Ultrasonic Imaging* **13** (1991), no. 2, p. 111 – 134.

[178] J. Ophir, S. K. Alam, B. S. Garra, F. Kallel, E. E. Konofagou, T. Krouskop, C. R. B. Merritt, R. Righetti, R. Souchon, S. Srinivasan & T. Varghese – "Elastography: Imaging the elastic properties of soft tissues with ultrasound", *Journal of Medical Ultrasonics* **29** (2002), no. 4, p. 155–171.

[179] K. J. Parker, M. M. Doyley & D. J. Rubens – "Imaging the elastic properties of tissue: the 20 year perspective", *Physics in Medicine and Biology* **56** (2011), no. 1, p. R1.

[180] B. Perthame & L. Vega – "Morrey-Campanato estimates for Helmholtz equations", *J. Funct. Anal.* **164** (1999), no. 2, p. 340–355.

[181] G. Popov & G. Vodev – "Resonances near the real axis for transparent obstacles", *Comm. Math. Phys.* **207** (1999), no. 2, p. 411–438.

[182] A. Pulkkinen, B. T. Cox, S. R. Arridge, H. Goh, J. P. Kaipio & T. Tarvainen – "Direct estimation of optical parameters from photoacoustic time series in quantitative photoacoustic tomography", *IEEE Transactions on Medical Imaging* **PP** (2016), no. 99, p. 1–1.

[183] A. Pulkkinen, B. T. Cox, S. R. Arridge, J. P. Kaipio & T. Tarvainen – "A Bayesian approach to spectral quantitative photoacoustic tomography", *Inverse Problems* **30** (2014), no. 6, p. 065012.

[184] T. Radó – "Aufgabe 41", *Jahresber. Deutsch. Math.-Verein* **35** (1926), p. 49.

[185] H. Reimann – "Functions of bounded mean oscillation and quasiconformal mappings", *Commentarii Mathematici Helvetici* **49** (1974), no. 1, p. 260–276 (English).

[186] K. Ren, H. Gao & H. Zhao – "A hybrid reconstruction method for quantitative PAT", *SIAM J. Imaging Sci.* **6** (2013), no. 1, p. 32–55.

[187] L. Robbiano – "Sur les zéros de solutions d'inégalités différentielles elliptiques", *Comm. Partial Differential Equations* **12** (1987), no. 8, p. 903–919.

[188] A. Rosenthal, D. Razansky & V. Ntziachristos – "Quantitative op-toacoustic signal extraction using sparse signal representation", *Medical Imaging, IEEE Transactions on* **28** (2009), no. 12, p. 1997–2006.

[189] D. L. Russell – "Controllability and stabilizability theory for linear par-tial differential equations: recent progress and open questions", *SIAM Rev.* **20** (1978), no. 4, p. 639–739.

[190] L. Sandrin, M. Tanter, S. Catheline & M. Fink – "Shear modulus imaging with 2-d transient elastography", *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control* **49** (2002), no. 4, p. 426–435.

[191] T. Saratoon, T. Tarvainen, B. T. Cox & A. S. R. – "A gradient-based method for quantitative photoacoustic tomography using the radiative transfer equation", *Inverse Problems* **29** (2013), no. 7, p. 075006.

[192] Scherzer (ed.) – *Handbook of Mathematical Methods in Imaging*, 2nd ed., vol. I,II,III, Springer, July 2015.

[193] F. Schulz – *Regularity theory for quasilinear elliptic systems and Monge-Ampère equations in two dimensions*, Lecture Notes in Mathematics, vol. 1445, Springer-Verlag, Berlin, 1990.

[194] G. Scott, M. Joy, R. Armstrong & R. Henkelman – "Measurement of nonuniform current density by magnetic resonance", *Medical Imaging, IEEE Transactions on* **10** (1991), no. 3, p. 362–374.

[195] J. K. Seo, D. Kim, J. Lee, O. I. Kwon, S. Z. K. Sajib & E. J. Woo – "Elec-trical tissue property imaging using MRI at dc and Larmor frequency", *Inverse Problems* **28** (2012), no. 8, p. 084002.

[196] J. K. Seo & E. J. Woo – "Magnetic resonance electrical impedance to-mography (MREIT)", *SIAM review* **53** (2011), no. 1, p. 40–68.

[197] C. G. Simader – *On Dirichlet's boundary value problem*, Lecture Notes in Mathematics, Vol. 268, Springer-Verlag, Berlin-New York, 1972, An L^p-theory based on a generalization of Gårding's inequality.

[198] E. Somersalo, D. Isaacson & M. Cheney – "A linearized inverse bound-ary value problem for Maxwell's equations", *J. Comput. Appl. Math.* **42** (1992), no. 1, p. 123–136.

[199] P. Stefanov & G. Uhlmann – "Thermoacoustic tomography with variable sound speed", *Inverse Problems* **25** (2009), no. 7, p. 075011.

[200] P. Stefanov & Y. Yang – "Multiwave tomography in a closed domain: averaged sharp time reversal", *Inverse Problems* **31** (2015), no. 6, p. 065007.

[201] E. M. Stein & R. Shakarchi – *Complex analysis,* Princeton Lectures in Analysis, II, Princeton University Press, Princeton, NJ, 2003.

[202] J. Sylvester & G. Uhlmann – "A global uniqueness theorem for an inverse boundary value problem", *Ann. of Math. (2)* **125** (1987), no. 1, p. 153–169.

[203] M. Tanter, J. Bercoff, A. Athanasiou, T. Deffieux, J.-L. Gennisson, G. Montaldo, M. Muller, A. Tardivon & M. Fink – "Quantitative assessment of breast lesion viscoelasticity: Initial clinical results using supersonic shear imaging", *Ultrasound in Medicine and Biology* **34** (2008), no. 9, p. 1373–1386 (English).

[204] L. Tartar – *The general theory of homogenization,* Lecture Notes of the Unione Matematica Italiana, vol. 7, Springer-Verlag, Berlin; UMI, Bologna, 2009, A personalized introduction.

[205] F. Triki – "Uniqueness and stability for the inverse medium problem with internal data", *Inverse Problems* **26** (2010), no. 9, p. 095014.

[206] G. M. Troianiello – *Elliptic differential equations and obstacle problems,* The University Series in Mathematics, Plenum Press, New York, 1987.

[207] G. Uhlmann – "Electrical impedance tomography and Calderón's problem", *Inverse Problems* **25** (2009), no. 12, p. 123011.

[208] K. Wang & M. A. Anastasio – "Photoacoustic and Thermoacoustic Tomography: Image Formation Principles", in *Handbook of Mathematical Methods in Imaging* (O. Scherzer, ed.), Springer New York, 2011, p. 781–815.

[209] G. N. Watson – *A treatise on the theory of Bessel functions,* Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.

[210] C. Weber – "Regularity theorems for Maxwell's equations", *Math. Methods Appl. Sci.* **3** (1981), no. 4, p. 523–536.

[211] N. Weck – "Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries", *J. Math. Anal. Appl.* **46** (1974), p. 410–437.

[212] T. Widlak & O. Scherzer – "Hybrid tomography for conductivity imaging", *Inverse Problems* **28** (2012), no. 8, p. 084008.

[213] T. H. Wolff – "Recent work on sharp estimates in second-order elliptic unique continuation problems", *J. Geom. Anal.* **3** (1993), no. 6, p. 621–650.

[214] E. J. Woo, S. Y. Lee & C. W. Mun – "Impedance tomography using internal current density distribution measured by nuclear magnetic resonance", *Proc. SPIE* **2299** (1994), p. 377–385.

[215] J. C. Wood – "Lewy's theorem fails in higher dimensions", *Math. Scand.* **69** (1991), no. 2, p. 166 (1992).

[216] L. Yao, G. Guo & H. Jiang – "Quantitative microwave-induced thermoacoustic tomography", *Medical Physics* **37** (2010), no. 7, p. 3752–3759.

[217] H.-M. Yin – "Regularity of weak solution to Maxwell's equations and applications to microwave heating", *J. Differential Equations* **200** (2004), no. 1, p. 137–161.

[218] H. Zhang & L. V. Wang – "Acousto-electric tomography", *Proc. SPIE. Photons Plus Ultrasound: Imaging and Sensing* **5320** (2004), p. 145–149.

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Lectures on Elliptic Methods for Hybrid Inverse Problems

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In recent years, several new imaging modalities have been developed in order to be able to detect physical parameters simultaneously at a high spatial resolution and with a high sensitivity to contrast. These new approaches typically rely on the interaction of two physical imaging methods, and the corresponding mathematical models are the so-called hybrid, or coupled-physics, inverse problems. The combination of two physical modalities poses new mathematical challenges: the analysis of this new class of inverse problems requires the use of various mathematical tools, often of independent interest. This book intends to provide a first comprehensive course on some of these tools (mainly related to elliptic partial differential equations) and on their applications to hybrid inverse problems.

For certain topics, such as the observability of the wave equation, the generalisation of the Radó-Kneser-Choquet Theorem to the conductivity equation, complex geometrical optics solutions and the Runge approximation property, we review well-known results. The material is presented with a clear focus on the intended applications to inverse problems. On other topics, including the regularity theory and the study of small-volume perturbations for Maxwell's equations, scattering estimates for the Helmholtz equation and the study of non-zero constraints for solutions of certain PDE, we discuss several new results. We then show how all these tools can be applied to the analysis of the parameter reconstruction for some hybrid inverse problems: Acousto-Electric tomography, Current Density Impedance Imaging, Dynamic Elastography, Thermoacoustic and Photoacoustic Tomography.

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